

Stability of a Leap-Frog Discontinuous Galerkin Method for Time-Domain Maxwell's Equations in Anisotropic Materials

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Abstract. In this work we discuss the numerical discretization of the time-dependent Maxwell's equations using a fully explicit leap-frog type discontinuous Galerkin method. We present a sufficient condition for the stability and error estimates, for cases of typical boundary conditions, either perfect electric, perfect magnetic or first order Silver-Müller. The bounds of the stability region point out the influence of not only the mesh size but also the dependence on the choice of the numerical flux and the degree of the polynomials used in the construction of the finite element space, making possible to balance accuracy and computational efficiency. In the model we consider heterogeneous anisotropic permittivity tensors which arise naturally in many applications of interest. Numerical results supporting the analysis are provided.

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Key words: Maxwell's equations, fully explicit leap-frog discontinuous Galerkin method, stability.

1 Introduction

Maxwell's equations are a fundamental set of partial differential equations which describe electromagnetic wave interactions with materials. The advantages of using discontinuous Galerkin time domain (DGTD) methods on the simulation of electromagnetic waves propagation, when compared with classical finite-difference time-domain methods, finite volume time domain methods or finite element time domain methods, have been reported by several authors (see *e.g.* [9] and references therein cited for an overview). DGTD methods gather many desirable features such as being able to achieve

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high-order accuracy and easily handle complex geometries. Moreover, they are suitable for parallel implementation on modern multi-graphics processing units. Local refinement strategies can be incorporated due to the possibility of considering irregular meshes with hanging nodes and local spaces of different orders.

The staggered leapfrog time-stepping algorithm is a popular choice for time domain Maxwell's equations (*e.g.* [1, 9, 21]) due to its simplicity, as it does not require to save in memory previous states, accuracy and robustness.

Despite the relevance of the anisotropic case in applications (*e.g.* [4, 14, 24]), most of the formulation of the DGTD methods presented in the literature are restricted to isotropic materials [11, 12, 16]. Motivated by our application of interest described in [2, 20], in the present paper we consider a model with a heterogeneous anisotropic permittivity tensor. The treatment of anisotropic materials within a DGTD framework was discussed for instance in [9] (with central fluxes) and in [13] (with upwind fluxes). The stability analysis of DGTD methods for Maxwell's equations was considered in [9], where the scheme that is defined with the central fluxes leads to a locally implicit time method in the case of Silver-Müller absorbing boundary conditions, and [15], where the scheme is defined with the upwind fluxes leading to an implicit method. Our derivation extends the results in [9] and [15] to a fully explicit in time method for both cases, central fluxes and upwind fluxes.

We consider the formulation in two dimensions as well as an extension to a three dimensional problem and we combine the nodal DG method [11] for the integration in space, considering both central and upwind fluxes, with an explicit leap-frog type method for the time integration. We present a rigorous proof of stability showing the influence of the mesh size, the choice of the numerical flux and choice of the degree of the polynomials used in the construction of the finite element space and the boundary conditions, which can be either perfect electric, perfect magnetic or first order Silver-Müller.

This paper consists in six sections after this introduction. In Section 2, we state the problem and in Section 3 we describe the formulation of the numerical method for the two-dimensional problem. In Section 4 we derive stability and convergence results for the method described in the previous section. We illustrate the theoretical results with numerical examples in Section 5. In the last section we extend the stability results to the three dimensional case.

2 The governing equations

The electromagnetic field consists of coupled electric and magnetic fields, known as electric field intensity, E , and magnetic induction, B . The effects of these two fundamental fields on matter can be characterized by the electric displacement and the magnetic field intensity vectors, frequently denoted by D and H , respectively. The knowledge of the material properties can be used to derive a useful relation between D and E and between B and H . Here we will consider the constitutive relations of the form $D = \epsilon E$ and $B = \mu H$,