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A Conservative Parallel Iteration Scheme for Nonlinear Diffusion Equations on Unstructured Meshes

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Abstract. In this paper, a conservative parallel iteration scheme is constructed to solve nonlinear diffusion equations on unstructured polygonal meshes. The design is based on two main ingredients: the first is that the parallelized domain decomposition is embedded into the nonlinear iteration; the second is that prediction and correction steps are applied at subdomain interfaces in the parallelized domain decomposition method. A new prediction approach is proposed to obtain an efficient conservative parallel finite volume scheme. The numerical experiments show that our parallel scheme is second-order accurate, unconditionally stable, conservative and has linear parallel speed-up.

AMS subject classifications: 65M08, 65E05

Key words: Diffusion equations, conservation, parallel scheme, unstructured mesh.

1 Introduction

Nonlinear diffusion problems appear in many applications such as radiation hydrodynamics problems in high energy density physics, and pollutant transport in subsurface geological formations. The computation cost is always very expensive to solve these equations numerically, so it is necessary to solve them on parallel supercomputers. As we know, the classical explicit scheme has natural parallelism and can be implemented directly on parallel computers, but it requires severe restrictions on the time step to ensure the stability. The implicit scheme does not have the stability restrictions, however, its

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parallelism is limited and cannot be implemented directly on parallel computers. Especially, it results in a large global nonlinear algebraic system at each time step, and solving this system on parallel computers needs a lot of global communications. For the global communication is a main bottle-neck to get high parallel efficiency, it is necessary to modify the implicit scheme into a parallel scheme which can avoid the global communication.

The parallelized domain decomposition method is one of powerful numerical techniques on parallel computations [12]. In [1], a finite difference domain decomposition algorithm is proposed, in which an explicit scheme on coarse meshes is used at inner interfaces. This algorithm can take larger time step than the classical explicit scheme, but it is only conditionally stable. In [4], an unconditionally stable domain decomposition method is presented, the accuracy of which is $\mathcal{O}(\tau^{\frac{1}{2}}+h)$, where τ and h denote the time step and the mesh size respectively. The accuracy of this method is improved in [6], but the spatial accuracy remains first order. Using the prediction and correction techniques at the interfaces of subdomains, some unconditionally stable parallel difference schemes with second-order accuracy are constructed in [7, 10, 11, 14, 16]. However, these parallel schemes are not conservative since the discrete fluxes are not equal at the interfaces of subdomains.

Practical computations show that numerical errors from non-conservative schemes are accumulated dramatically through a large number of time steps, which leads to the numerical solution is far from the exact solution, especially on highly distorted meshes or in the region where the physical quantity varies intensively. Therefore, it is very important to enforce the conservation in the construction of the parallel scheme. In [2,3], the explicit/implicit conservative domain decomposition procedures are presented, in which the fluxes at the inner interfaces are calculated by the values from the previous time step. These schemes are conditionally stable, and the rigorous stability and convergence analysis can be found in [15]. A conservative parallel scheme for the linear diffusion equation on one-dimensional meshes is presented in [18], in which the unconditional stability and the second-order accuracy of the scheme are proved. However, when this scheme is directly extended to two-dimensional problems, it becomes conditionally stable. In [17], a conservative domain decomposition procedure is proposed for nonlinear diffusion problems on quadrilateral (structured) meshes. In this method, the values from the previous nonlinear iteration step are adopted as the Dirichlet boundary condition of subdomain problems at the inner interfaces, and then a correction step is implemented to ensure the conservation at the inner interfaces. Numerical experiments show that this parallel scheme is unconditionally stable, second-order accurate and conservative.

In practice a good parallel diffusion scheme is expected to satisfy the following requirements: (1) unconditionally stable, (2) second-order accurate, (3) conservative, (4) highly parallel, (5) suitable to various meshes, and no restrictions for the interface shape between subdomains. We point out that the last requirement comes from some applications such as the complex flow in Lagrangian radiation hydrodynamics problems, and pollutant transport in complex regions. For the multi-material radiation hydrodynamics problems with large variation of flow field, the Lagrangian computational meshes are