

## Two-Grid Method for Miscible Displacement Problem by Mixed Finite Element Methods and Mixed Finite Element Method of Characteristics

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**Abstract.** The miscible displacement of one incompressible fluid by another in a porous medium is governed by a system of two equations. One is elliptic form equation for the pressure and the other is parabolic form equation for the concentration of one of the fluids. Since only the velocity and not the pressure appears explicitly in the concentration equation, we use a mixed finite element method for the approximation of the pressure equation and mixed finite element method with characteristics for the concentration equation. To linearize the mixed-method equations, we use a two-grid algorithm based on the Newton iteration method for this full discrete scheme problems. First, we solve the original nonlinear equations on the coarse grid, then, we solve the linearized problem on the fine grid used Newton iteration once. It is shown that the coarse grid can be much coarser than the fine grid and achieve asymptotically optimal approximation as long as the mesh sizes satisfy  $h = H^2$  in this paper. Finally, numerical experiment indicates that two-grid algorithm is very effective.

**AMS subject classifications:** 35M13, 65M12

**Key words:** Two-grid method, miscible displacement problem, mixed finite element, characteristic finite element method.

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## 1 Introduction

We consider the miscible displacement of one incompressible fluid by another in a reservoir  $\Omega \subset \mathbb{R}^2$  of unit thickness. The nonlinear coupled system of equations that describes

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the pressure  $p(x,t)$  and the concentration  $c(x,t)$  of one of the fluids is given by

$$\phi \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c - \nabla \cdot (D \nabla c) = f(c), \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = q, \quad (1.2)$$

$$\mathbf{u} = -a(c) \nabla p, \quad (1.3)$$

where  $x \in \Omega$ ,  $t \in J = [0, T]$  and  $a(c) = a(x, c) = \frac{k(x)}{\mu(c)}$ .  $k(x)$  is the permeability of the porous rock,  $\mu(c)$  is the viscosity of the fluid mixture,  $\mathbf{u}(x, t)$  is the Darcy velocity of the mixture,  $q(x, t)$  represents the flow rate at wells and  $q^+$  is the positive part of the  $q$ .  $f(c)$  may be nonlinear function [17].  $c_0(x)$  is the initial concentration,  $\phi(x)$  is the porosity of the rock, and  $D(\mathbf{u})$  is the coefficient of molecular diffusion and mechanical dispersion of one fluid into the other and it is the  $2 \times 2$  matrix,

$$D = \phi [d_m I + |\mathbf{u}| (d_l E(\mathbf{u}) + d_t E^\perp(\mathbf{u}))],$$

where  $E(\mathbf{u}) = \frac{u_i u_j}{|\mathbf{u}|^2}$  and  $E^\perp = I - E$ ,  $d_m$  is the molecular diffusion,  $d_l$  and  $d_t$  are, respectively, the longitudinal and transverse dispersion coefficients. For convenience, we assume that  $D = \phi d_m I$  implies only the molecular diffusion and not the dispersion in this paper.

The system is subjected to boundary conditions:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad x \in \partial\Omega, \quad t \in J, \quad (1.4)$$

$$D \nabla c \cdot \mathbf{n} = 0, \quad x \in \partial\Omega, \quad t \in J, \quad (1.5)$$

and an initial condition

$$c(x, 0) = c_0(x), \quad x \in \Omega. \quad (1.6)$$

Two-phase flow and transportation of fluids in porous media play an important role in both theoretic and applicative aspects. Petroleum engineers have been interested in efficient oil exploitation and the hydrologists have been concerned about improving the utilization of groundwater resource for a long time. Chen and Ewing [7] have studied the mathematic theory related to these models.

In the past thirty years, numerical approximations have received a great deal of attention for the miscible displacement of one incompressible fluid by another in a reservoir [10–12, 14–16, 23, 24]. Ewing and Wheeler [10] use Galerkin methods for both continuous time and full discrete approximation. Douglas and his coauthors proposed a finite element method based on the use of an elliptic mixed finite element method to approximate the pressure  $p$  and the velocity  $\mathbf{u}$  and a parabolic Galerkin method to approximate the concentration [16]. It is not surprising that a mixed method [18, 19], Since only the velocity and not the pressure appears explicitly in Eq. (1.1) for the concentration.

In realistic displacements  $D$  is quite small for the concentration equation, so that (1.1) for  $c$  is strongly convection-dominated. Standard Galerkin methods produce unacceptable nonphysical and oscillations in the concentration approximations, thus, Douglas