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A Full Space-Time Convergence Order Analysis of Operator Splittings for Linear Dissipative Evolution Equations

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Abstract. The Douglas-Rachford and Peaceman-Rachford splitting methods are common choices for temporal discretizations of evolution equations. In this paper we combine these methods with spatial discretizations fulfilling some easily verifiable criteria. In the setting of linear dissipative evolution equations we prove optimal convergence orders, simultaneously in time and space. We apply our abstract results to dimension splitting of a 2D diffusion problem, where a finite element method is used for spatial discretization. To conclude, the convergence results are illustrated with numerical experiments.

AMS subject classifications: 65J08, 65M12, 65M60

Key words: Douglas/Peaceman-Rachford schemes, full space-time discretization, dimension splitting, convergence order, evolution equations, finite element methods.

1 Introduction

We consider the linear evolution equation

$$\dot{u} = Lu = (A+B)u, \quad u(0) = \eta, \tag{1.1}$$

where L is an unbounded, dissipative operator. Such equations are commonly encountered in the natural sciences, e.g. when modeling advection-diffusion processes. Splitting methods are widely used for temporal discretizations of evolution equations. The competitiveness of these methods is attributed to their separation of the flows generated by A and B. In many applications these separated flows can be more efficiently evaluated

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than the flow related to L; a prominent example being that of dimension splitting. We refer to [9,14,18] for general surveys.

In the present paper we consider the combined effect of temporal and spatial discretization when the former is given by either the Douglas-Rachford scheme

$$S=(I-kB)^{-1}(I-kA)^{-1}(I+k^2AB), \tag{1.2}$$

or the Peaceman-Rachford scheme

$$S=\Big(I-\frac{k}{2}B\Big)^{-1}\Big(I+\frac{k}{2}A\Big)\Big(I-\frac{k}{2}A\Big)^{-1}\Big(I+\frac{k}{2}B\Big). \tag{1.3}$$

Here, S denotes the operator that takes a single time step of size k. Thus $S^n\eta$ constitutes a temporal splitting approximation at time $t=nk>0$ of the solution $u(t)=\mathrm{e}^{tL}\eta$ of Eq. (1.1). The first order Douglas-Rachford scheme can be constructed as a modification of the simple Lie splitting resulting in an advantageous error structure. An exposition is given in [12]. The Peaceman-Rachford scheme was introduced in [21] as a dimension splitting of the heat equation. A temporal convergence order analysis of the scheme for linear evolution equations is given in [11], which also features an application to dimension splitting. Convergence orders in time are proven in [6,10] for nonlinear operators B under various assumptions on the nonlinearity. See also [23] for further stability considerations.

In the general setting the operators A, B, and L are infinite dimensional. Therefore, to define an algorithm that can be implemented, any numerical method must replace these operators, that is, a spatial discretization is needed. In our abstract analysis, we consider any spatial discretization fulfilling some assumptions ensuring convergence for the stationary problem. When both a temporal and a spatial discretization has been employed to Eq. (1.1) we refer to it as being *fully discretized*. Under similar assumptions to ours, convergence orders are proven in [5,25] for full discretizations where implicit Euler or Crank-Nicholson is used as temporal discretization.

Our abstract analysis is applied to dimension splitting combined with a finite element method. As is usually done in practice, we consider spatial discretizations where the finite element matrices are constructed with the help of numerical quadrature schemes. We will refer to these discretization methods as *quadrature finite element methods*. Convergence order analyses for quadrature finite element methods are carried out for linear elliptic PDEs in [3,4,24,25] and when they are used as spatial semi-discretizations for a linear parabolic problem in [22]. Full discretizations of a nonlinear parabolic PDE, where the spatial discretization is given by quadrature finite elements, are considered in [19]. There, convergence orders are derived when explicit Euler, implicit Euler or a modified Crank-Nicholson method is used as temporal discretization.

Earlier results about the combined effects of splitting methods and spatial discretizations include the recent paper [2]. There, convergence without orders is proven for full discretizations when exponential splittings are used for temporal discretization of the abstract evolution equation (1.1). Full space-time convergence studies for semi-implicit methods applied to various semilinear evolution equations can be found in [1,16,25]. A