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Entropy Stable Scheme on Two-Dimensional Unstructured Grids for Euler Equations

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Abstract. We propose an entropy stable high-resolution finite volume scheme to approximate systems of two-dimensional symmetrizable conservation laws on unstructured grids. In particular we consider Euler equations governing compressible flows. The scheme is constructed using a combination of entropy conservative fluxes and entropy-stable numerical dissipation operators. High resolution is achieved based on a linear reconstruction procedure satisfying a suitable sign property that helps to maintain entropy stability. The proposed scheme is demonstrated to robustly approximate complex flow features by a series of benchmark numerical experiments.

AMS subject classifications: 65M08, 35L65, 35Q31

Key words: Entropy stability, Euler equations, unstructured grids, sign property, higher-order accuracy.

1 Introduction

Systems of conservation laws are encountered in numerous fields of science and engineering. Examples include the shallow water equations of oceanography, the Euler equations of aerodynamics and the MHD equations of plasma physics. In two space dimensions, a generic system of conservation laws is given by

$$\begin{aligned}&\partial_t \mathbf{U}+\partial_x \mathbf{f}_1(\mathbf{U})+\partial_y \mathbf{f}_2(\mathbf{U})=0 \qquad \forall\, \mathbf{x}=(x,y)\in\mathbb{R}^2,\ t\in\mathbb{R}^+\\ &\mathbf{U}(\mathbf{x},0)=\mathbf{U}_0(\mathbf{x}) \qquad \forall\, \mathbf{x}\in\mathbb{R}^2.\end{aligned} \tag{1.1}$$

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In the above equations, the vector of conserved variables is denoted by $\mathbf{U}:\mathbb{R}^2\times\mathbb{R}^+\to\mathbb{R}^n$, $\mathbf{f}_1,\mathbf{f}_2$ are the Cartesian components of the flux vector and $\mathbf{U}_0$ is the prescribed initial condition. In particular, for the two-dimensional compressible Euler equations, we have

$$\mathbf{U}=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ E\end{pmatrix},\qquad \mathbf{f}_1(\mathbf{U})=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(E+p)\end{pmatrix},\qquad \mathbf{f}_2(\mathbf{U})=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(E+p)\end{pmatrix},\tag{1.2}$$

where ρ, $\mathbf{u}=(u,v)^\top$ and p denote the fluid density, velocity and pressure, respectively. The quantity E is the total energy per unit volume

$$E=\rho\left(\frac{1}{2}(u^2+v^2)+e\right),\tag{1.3}$$

where e is the specific internal energy given by a caloric equation of state, $e=e(\rho,p)$. In this work, we take the equation of state for ideal gas as given by

$$e=\frac{p}{(\gamma-1)\rho}\tag{1.4}$$

with $\gamma=c_p/c_v$ denoting the ratio of specific heats.

It is well known that solutions to systems of conservation laws can develop discontinuities, such as shock waves and contact discontinuities, in finite time even when the initial data is smooth [11]. Hence, the solutions of systems of conservation laws are interpreted in a weak (distributional) sense. However, these weak solutions are not necessarily unique, and must be supplemented with additional conditions, known as the *entropy conditions*, in order to single out a physically relevant solution. Assume that for the system (1.1), there exists a convex function $\eta:\mathbb{R}^n\to\mathbb{R}$ and functions $q_i:\mathbb{R}^n\to\mathbb{R}$, $i=1,2$ such that

$$q_i'(\mathbf{U})=\eta'(\mathbf{U})^\top\mathbf{f}_i'(\mathbf{U}),\qquad i=1,2.\tag{1.5}$$

The function η is known as an *entropy function*, while q_1, q_2 are the *entropy flux functions*. Additionally, $\mathbf{V}=\eta'(\mathbf{U})$ is called the (vector of) *entropy variables*. Multiplying (1.1) by $\mathbf{V}^\top$ results in the following additional conservation law for smooth solutions:

$$\partial_t\eta(\mathbf{U})+\partial_x q_1(\mathbf{U})+\partial_y q_2(\mathbf{U})=0.\tag{1.6}$$

The entropy condition states that weak solutions should satisfy the entropy inequality

$$\partial_t\eta(\mathbf{U})+\partial_x q_1(\mathbf{U})+\partial_y q_2(\mathbf{U})\leqslant 0,\tag{1.7}$$

which is understood in the sense of distributions.

The convexity of $\eta(\mathbf{U})$ ensures the existence of a one-to-one mapping between $\mathbf{U}$ and $\mathbf{V}$, thus allowing the change of variables $\mathbf{U}=\mathbf{U}(\mathbf{V})$. The hyperbolic system (1.1) is *symmetrized* when written in terms of the entropy variables. In other words, for the transformed system

$$\partial_{\mathbf{V}}\mathbf{U}\partial_t\mathbf{V}+\partial_{\mathbf{V}}\mathbf{f}_1\partial_x\mathbf{V}+\partial_{\mathbf{V}}\mathbf{f}_2\partial_y\mathbf{V}=0$$