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Probabilistic High Order Numerical Schemes for Fully Nonlinear Parabolic PDEs

Tao Kong[1], Weidong Zhao[1] and Tao Zhou[2,*]

[1] *School of Mathematics & Finance Institute, Shandong University, Jinan 250100, China.*
[2] *LSEC, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.*

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Abstract. In this paper, we are concerned with probabilistic high order numerical schemes for Cauchy problems of fully nonlinear parabolic PDEs. For such parabolic PDEs, it is shown by Cheridito, Soner, Touzi and Victoir [4] that the associated exact solutions admit probabilistic interpretations, i.e., the solution of a fully nonlinear parabolic PDE solves a corresponding second order forward backward stochastic differential equation (2FBSDEs). Our numerical schemes rely on solving those 2FBSDEs, by extending our previous results [W. Zhao, Y. Fu and T. Zhou, SIAM J. Sci. Comput., 36 (2014), pp. A1731-A1751.]. Moreover, in our numerical schemes, one has the flexibility to choose the associated forward SDE, and a suitable choice can significantly reduce the computational complexity. Various numerical examples including the HJB equations are presented to show the effectiveness and accuracy of the proposed numerical schemes.

AMS subject classifications: 60H35, 65H20, 65H30

Key words: Fully nonlinear parabolic PDEs, second order FBSDEs, probabilistic interpretations, probabilistic numerical schemes.

1 Introduction

The paper is concerned with probabilistic numerical schemes for solving nonlinear parabolic PDEs in the following form:

$$\begin{cases} u_t+F(t,x,u,Du,D^2u)=0, & (t,x)\in[0,T)\times\mathbb{R}^m, \\ u(T,x)=g(x), & x\in\mathbb{R}^m, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `vision.kt@gmail.com` (T. Kong), `wdzhao@sdu.edu.cn` (W. Zhao), `tzhou@lsec.cc.ac.cn` (T. Zhou)

where $u(\cdot,\cdot)$ is a map from $[0,T]\times\mathbb{R}^m\to\mathbb{R}$; $Du(x)$ and $D^2u(x)$ stand for the gradient and the Hessian matrix of u with respect to x, respectively. The nonlinear operator F is a map $[0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^m\times\mathcal{S}^m\to\mathbb{R}$ and $g:\mathbb{R}^m\to\mathbb{R}$ is the terminal condition. The PDEs (1.1) are called fully nonlinear if the operator F is nonlinear with respect to the highest order derivatives D^2u.

Recently, there has been a great interest to derive probabilistic interpretations for solutions of PDEs. Pioneer work is due to Pardox and Peng [17], where they show that the quasi-linear parabolic PDE is associated to a Markovian Backward SDE due to the nonlinear Feynman-Kac formula introduced by Pardoux and Peng [17]. Extensions to more general parabolic PDEs can be found in [2,20]. To link fully nonlinear parabolic PDEs and backward SDEs, an recent work by Cheridito, Soner, Touzi and Victoir [4] introduced a notion of second order forward backward SDEs (2FBSDEs). They show that the solution of the fully nonlinear parabolic PDE solves a corresponding 2FBSDEs. We note that the G-expectation, a nonlinear expectation introduced by Peng [19] also deals with this issue.

Based on these probabilistic interpretations, one can derive the so called probabilistic numerical schemes for solving PDEs. In the quasi-linear case, the PDE is associated to a Markovian Backward SDE due to the nonlinear Feynman-Kac formula introduced by Pardoux and Peng [17]. One can refer to [2, 6, 7] and references therein for probabilistic numerical schemes, and to [1, 3, 5, 8, 11, 15, 16, 24–28] for numerical schemes for FBSDEs. There have also been numerous publications on the subject and the schemes have been extended to more general BSDEs, e.g. reflected BSDEs which is appropriate for pricing and hedging American options.

However, there are only a few work on 2FBSDEs [13, 22] and fully non-linear PDEs [9, 12, 23]. Moreover, existing work on fully non-linear PDEs aims at designing efficient schemes for high dimensional PDEs, however, the convergence rates are not satisfactory. In particular, we mention the work [12], where a numerical example for a 12-dimensional coupled FBSDE is reported, and it is shown by numerical test that the numerical method converges with order 1. Also, in [13], multistep schemes were proposed to solve 2FBSDEs, and high order convergence rates were obtained, however, only for low dimensional examples. We also note that people in the numerical PDEs community are paying more and more attention to the numerical approaches for fully nonlinear PDEs [10].

In this work, we aim at designing high order probabilistic numerical schemes for Cauchy problems of fully nonlinear parabolic PDEs. Our numerical schemes rely on solving those equivalent 2FBSDEs, by extending our previous results in [26], where the Euler-type method were used for the forward SDE, and highly accurate multistep method were used to approximate the derivatives derived from the backward stochastic differential equation in FBSDEs. The Euler method used to solve the forward SDE dramatically reduces the computational complexity. We show that in our framework one has the flexibility to choose the associated forward SDE, and a suitable choice can significantly reduce the computational complexity. Various numerical examples including the HJB equations are presented to show effectiveness and accuracy of the proposed numerical schemes.

The rest of the paper is organized as follows. In Section 2, we introduce some prelim-