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Piecewise Polynomial Mapping Method and Corresponding WENO Scheme with Improved Resolution

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Abstract. The method of mapping function was first proposed by Henrick et al. [J. Comput. Phys. 207:542-547 (2005)] to adjust nonlinear weights in [0,1] for the fifth-order WENO scheme, and through which the requirement of convergence order is satisfied and the performance of the scheme is improved. Different from Henrick's method, a concept of piecewise polynomial function is proposed in this study and corresponding WENO schemes are obtained. The advantage of the new method is that the function can have a gentle profile at the location of the linear weight (or the mapped nonlinear weight can be close to its linear counterpart), and therefore is favorable for the resolution enhancement. Besides, the function also has the flexibility of quick convergence to identity mapping near two endpoints of [0,1], which is favorable for improved numerical stability. The fourth-, fifth- and sixth-order polynomial functions are constructed correspondingly with different emphasis on aforementioned flatness and convergence. Among them, the fifth-order version has the flattest profile. To check the performance of the methods, the 1-D Shu-Osher problem, the 2-D Riemann problem and the double Mach reflection are tested with the comparison of WENO-M, WENO-Z and WENO-NS. The proposed new methods show the best resolution for describing shear-layer instability of the Riemann problem, and they also indicate high resolution in computations of double Mach reflection, where only these proposed schemes successfully resolved the vortex-pairing phenomenon. Other investigations have shown that the single polynomial mapping function has no advantage over the proposed piecewise one, and it is of no evident benefit to use the proposed method for the symmetric fifth-order WENO. Overall, the fifth-order piecewise polynomial and corresponding WENO scheme are suggested for resolution improvement.

AMS subject classifications: 65M06, 65M12

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1 Introduction

Following the introduction to the weighted essentially non-oscillatory (WENO) scheme [1], the subsequent efficient implementation [2] made the algorithm applicable to realistic problems. The weighting procedures and the smoothness indicator (IS) [2] eventually became a standard. After ten years of practices, WENO schemes especially the fifth-order version (WENO5) [2] have become one of the most popular high-order methods. Despite the success, some issues pertaining to WENO schemes were raised. It was Henrick et al. [3] who first pointed out that WENO5 failed to retain fifth-order accuracy at the critical point with $f_j' = 0$. They further proposed the necessary and sufficient conditions for a scheme to obtain fifth-order accuracy. As a remedy, Henrick et al. [3] proposed a carefully designed mapping function, through which the difference between the nonlinear weight and its linear counterpart will usually have the order of Δx^3. The corresponding scheme was called WENO-M, which preserves fifth-order accuracy at the critical point [3].

The performance of WENO-M was tested by cases such as the 1-D Shu-Osher problem at 400 points [3] and the 2-D double Mach reflection problem [4]. The improvement on resolution was clearly shown through the comparison with WENO5. However, Borges et al. [5] argued the improvement was not due to enhancement of the convergence order, but came more from the "assignment of larger weights to discontinuous stencils". Still conforming to the accuracy requirement as in Ref. [3], they proposed a new IS by using a term comprised of higher order derivatives. The corresponding scheme was called as WENO-Z, and preliminary tests showed its slightly better performance than that of WENO-M [3,4].

Focusing on revising IS, Ha et al. [4] proposed a new algorithm by combining numerical approximations of first- and second-order derivatives. Two considerations were noticed in their work, i.e., an undivided difference for derivative discretization and a parameter to control "the trade-off between the accuracies around the smooth region and discontinuity region". The so-called WENO-NS scheme has shown better resolution in the computations of double-Mach reflection and 2-D Riemann problems when compared with WENO-M and WENO-Z.

Resolution enhancement may run the risk of numerical instability. Our tests showed that when using WENO-NS, the computation of a 2-D supersonic flow around half cylinder at $M = 4$ blew up when the Steger-Warming scheme was used for flux splitting, to say nothing of tougher hypersonic cases. On the one hand, efforts continue on toward higher order and better resolution; on the other hand, schemes developed are expected to be robust and applicable for practical problems.

A procedure is proposed in this paper to improve resolution while preserving robustness. First, following the idea of mapping functions, specific piecewise polynomials of various orders are proposed, which are targeted toward resolution enhancement. The details are described in Section 2. Using the proposed methods, new fifth-order WENO schemes are obtained. Next, typical numerical tests are conducted in Section 3 with the comparison with WENO5, WENO-M, WENO-Z and WENO-NS. From computations of