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L_2 Convergence of the Lattice Boltzmann Method for One Dimensional Convection-Diffusion-Reaction Equations

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Abstract. Combining asymptotic analysis and weighted L_2 stability estimates, the convergence of lattice Boltzmann methods for the approximation of 1D convection-diffusion-reaction equations is proved. Unlike previous approaches, the proof does not require transformations to equivalent macroscopic equations.

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Key words: Lattice Boltzmann method, convection-diffusion-reaction equation, L_2 stability, L_2 convergence.

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1 Introduction

In this article, we consider a variant of the lattice Boltzmann method for the solution of the convection-diffusion-reaction equation (for example, see [3, 8, 16, 18, 19]). The practical validity of the method has been investigated through formal consistency analysis (Chapman-Enskog expansion or asymptotic expansion) and numerical convergence studies. On top of that, stability properties have been checked by numerical tests and investigation of spectral properties (von Neumann stability analysis). While these investigations are all important in their own right, convergence results add further confidence to the method.

At this point, it could be argued that the accessibility of consistency and stability results automatically entails convergence due to a general theorem of von Neumann. However, a detailed revision of the required prerequisites shows that the standard convergence theory is based on a more specific notion of consistency and stability than the one available for lattice Boltzmann methods. As far as consistency is concerned, this is due to the fact that the lattice Boltzmann equation approximates a singularly perturbed

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discrete velocity model with a coupling between discretization and perturbation parameters, leading to a more intricate behavior compared to discretizations of unperturbed differential equations. Also, the non-symmetry of the evolution matrix which describes the effect of a single lattice Boltzmann step renders the investigation of its eigenvalue structure useless for obtaining L_2-norm estimates which are employed in standard convergence proofs (of course, this does not reduce its importance in providing information on the stability of the solution for a fixed setting of the discretization parameters).

For these reasons, convergence proofs for lattice Boltzmann methods follow somewhat different strategies than in classical numerical analysis. As examples, we mention the work [7, 22] where convergence of lattice Boltzmann D1Q2 models is proved based on a technique which requires the equivalent moment equations. This technique is applicable for small systems in 1D but seems less efficient for higher dimensional cases, where the moment systems are much larger and cumbersome to set up.

Having all these aspects in mind, we can now formulate the aim of this paper: we want to present and advocate a different proof strategy which is based on the lattice Boltzmann equation itself and which extends to higher dimensions in a straightforward manner. As for the stability aspect, we will show that the results [1, 14] on stability of lattice Boltzmann methods for the Navier-Stokes equation can be extended to the case of the convection-diffusion-reaction equation. This stresses the universality of the approach. As a side effect, conditions on the discretization parameters (in the spirit of the CFL condition) will be found which guarantee convergence of the method in a particular case.

As for consistency, we employ asymptotic analysis and split the asymptotic description of the numerical solution into smooth and non-smooth parts. The smooth parts contain information on the solution of the convection-diffusion-reaction equation and possible errors which are of lower order in grid size and governed by known equations. The non-smooth part will be one order lower than the smooth error which allows us to reduce the smoothness assumptions on the data such as convection speed or reaction rate.

To demonstrate the basic ideas of the strategy, it is reasonable to restrict to 1D lattice Boltzmann schemes with simple periodic boundary conditions because, then, the proofs are particularly easy to display and examine. We hope that, in this way, the reader may understand the basic idea in each detail. Results for higher dimensional problems and non-periodic boundary conditions will appear in subsequent papers.

After presenting the basic equations in Section 2 and the consistency analysis in Section 3, stability results are summarized in Theorems 4.1, 4.2, and Theorem 5.3, where a priori estimates are obtained. The convergence results can be found in Section 5.

2 Lattice Boltzmann schemes in 1D

We consider a general one-dimensional convection-diffusion-reaction equation for some density $\rho:[0,T]\times\mathbb{R}\mapsto\mathbb{R}$,