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High Order Fixed-Point Sweeping WENO Methods for Steady State of Hyperbolic Conservation Laws and Its Convergence Study

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Abstract. Fixed-point iterative sweeping methods were developed in the literature to efficiently solve static Hamilton-Jacobi equations. This class of methods utilizes the Gauss-Seidel iterations and alternating sweeping strategy to achieve fast convergence rate. They take advantage of the properties of hyperbolic partial differential equations (PDEs) and try to cover a family of characteristics of the corresponding Hamilton-Jacobi equation in a certain direction simultaneously in each sweeping order. Different from other fast sweeping methods, fixed-point iterative sweeping methods have the advantages such as that they have explicit forms and do *not* involve inverse operation of nonlinear local systems. In principle, it can be applied in solving very general equations using any monotone numerical fluxes and high order approximations easily. In this paper, based on the recently developed fifth order WENO schemes which improve the convergence of the classical WENO schemes by removing slight post-shock oscillations, we design fifth order fixed-point sweeping WENO methods for efficient computation of steady state solution of hyperbolic conservation laws. Especially, we show that although the methods do *not* have linear computational complexity, they converge to steady state solutions much faster than regular time-marching approach by stability improvement for high order schemes with a forward Euler time-marching.

AMS subject classifications: 65M06, 65M12, 65N06, 65N12

Key words: Fixed-point sweeping methods, WENO methods, high order accuracy, steady state, hyperbolic conservation laws, convergence.

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1 Introduction

Steady state problems for hyperbolic partial differential equations (PDEs) are common mathematical models appearing in many applications, such as fluid mechanics, optimal control, differential games, image processing and computer vision, geometric optics, etc. Solution information of these boundary value problems propagates along characteristics starting from the boundary. Weighted essentially non-oscillatory (WENO) schemes are a popular class of high order numerical methods for spatial discretization of hyperbolic PDEs. They have the advantage of attaining uniform high order accuracy in smooth regions of the solution while maintaining sharp and essentially non-oscillatory transitions of discontinuities. WENO scheme was first constructed in [11] for a third-order accurate finite volume version. In [7], third- and fifth-order accurate finite difference WENO schemes in multi-space dimensions were constructed, with a general framework for the design of the smoothness indicators and nonlinear weights. To deal with complex domain geometries, WENO schemes on unstructured meshes were developed, e.g., see [6, 9, 12, 23, 24, 28].

A large nonlinear system is obtained after spatial discretization of a steady state hyperbolic PDE by a high order WENO scheme. It is still a challenging problem how to solve the large nonlinear system. There are at least two factors which may affect efficiency and robustness of computation. One is that a high order accurate shock capturing scheme such as a fifth order WENO scheme often suffers from difficulties in its convergence towards steady state solutions. In [21], A systematic study was carried out and discovered that slight post-shock oscillations actually cause this problem. A new smoothness indicator [21] and upwind-biased interpolation technique [20] have been developed to improve the convergence of fifth order WENO scheme for solving steady state of Euler systems. The other factor affecting the performance of computation is the iterative scheme designed for the nonlinear system. For a highly nonlinear system derived from high order WENO spatial discretization, one way is to solve it directly with Newton iterations or a more robust method such as the homotopy method [5]. Another way is to solve the large WENO system by fast sweeping technique [26]. Fast sweeping methods utilize alternating sweeping strategy to cover a family of characteristics in a certain direction simultaneously in each sweeping order. Coupled with the Gauss-Seidel iterations, these methods can achieve a fast convergence speed for computations of steady state solutions of hyperbolic PDEs. First order fast sweeping methods often achieve linear computational complexity for certain types of equations (e.g., see [4,13,14,27]). There are additional difficulties to design high order fast sweeping methods with linear computational complexity, including lack of monotonicity of numerical solutions, much more complicated local nonlinear equations, wider stencils which make alternating sweeping less effective, etc. High order WENO fast sweeping method was developed in [26]. An explicit strategy was designed to avoid directly solving very complicated local nonlinear equations derived from WENO discretizations. The method was extended to a fifth order version in [19] with accurate boundary treatment techniques. This explicit strat-