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Runge-Kutta Central Discontinuous Galerkin Methods for the Special Relativistic Hydrodynamics

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Abstract. This paper develops Runge-Kutta P^K-based central discontinuous Galerkin (CDG) methods with WENO limiter for the one- and two-dimensional special relativistic hydrodynamical (RHD) equations, $K=1,2,3$. Different from the non-central DG methods, the Runge-Kutta CDG methods have to find two approximate solutions defined on mutually dual meshes. For each mesh, the CDG approximate solutions on its dual mesh are used to calculate the flux values in the cell and on the cell boundary so that the approximate solutions on mutually dual meshes are coupled with each other, and the use of numerical flux will be avoided. The WENO limiter is adaptively implemented via two steps: the "troubled" cells are first identified by using a modified TVB minmod function, and then the WENO technique is used to locally reconstruct new polynomials of degree $(2K+1)$ replacing the CDG solutions inside the "troubled" cells by the cell average values of the CDG solutions in the neighboring cells as well as the original cell averages of the "troubled" cells. Because the WENO limiter is only employed for finite "troubled" cells, the computational cost can be as little as possible. The accuracy of the CDG without the numerical dissipation is analyzed and calculation of the flux integrals over the cells is also addressed. Several test problems in one and two dimensions are solved by using our Runge-Kutta CDG methods with WENO limiter. The computations demonstrate that our methods are stable, accurate, and robust in solving complex RHD problems.

AMS subject classifications: 76M10, 76M25, 76Y05, 76N15

Key words: Central discontinuous Galerkin method, WENO limiter, Runge-Kutta time discretization, relativistic hydrodynamics.

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1 Introduction

Relativistic fluid widely appears in nuclear physics, astrophysics, plasma physics, and other fields. For example, in the physical phenomena such as the formation of neutron stars and black holes and the high-speed jet, the local fluid velocity may be close to the speed of light, at this time the relativistic effect can not be neglected and the relativistic fluid dynamics (RHD) is needed. Because the RHD equations are more complicated, their theoretical analysis is impractical so that numerical simulation has become a primary and powerful way to study and understand the physical mechanisms in the RHDs.

The pioneering numerical work may date back to the finite difference code via artificial viscosity for the spherically symmetric general RHD equations in the Lagrangian coordinate [32, 33]. Wilson first attempted to solve multi-dimensional RHD equations in the Eulerian coordinate by using the finite difference method with the artificial viscosity technique [47]. Since 1990s, the numerical study of the RHDs began to attract considerable attention, and various modern shock-capturing methods with an exact or approximate Riemann solver have been developed for the RHD equations, the readers are referred to the early review articles [31, 46]. Some examples on existing methods, which are extensions of Godunov type shock capturing methods, are the upwind schemes based on local linearization [16, 17], the two shock approximation solvers [1, 12, 35], flux-vector splitting scheme [14], HLL (Harten-Lax-van Leer) schemes [15, 41], HLLC (Harten-Lax-van Leer-Contact) scheme [34], non-oscillatory essentially (ENO) schemes [13, 58], and kinetic schemes [21, 55] and so on. Recently the second author and his co-workers developed adaptive moving mesh method [18], derived the second-order accurate generalized Riemann problem (GRP) methods for the one- and two-dimensional RHD equations [56, 57], and the finite volume local evolution Galerkin scheme for two-dimensional RHD equations [48]. Later, the third-order accurate GRP scheme in [54] was extended to the one-dimensional RHD equations [53], and the direct Eulerian GRP scheme was developed for the spherically symmetric general relativistic hydrodynamics [49]. The physical-constraints-preserving (PCP) schemes were also studied for the special RHD equations recently. The high-order accurate PCP finite difference weighted essentially non-oscillatory (WENO) schemes and discontinuous Galerkin (DG) methods were proposed in [36, 50, 52]. Moreover, the set of admissible states and the PCP schemes of the ideal relativistic magnetohydrodynamics was studied for the first time in [51], where the importance of divergence-free fields was revealed in achieving PCP methods especially.

The DG methods have been rapidly developed in recent decades and become a kind of important methods in computational fluid dynamics. They are easy to achieve high order accuracy, suitable for parallel computing, and adapt to complex domain boundary. The DG method was first developed by Reed and Hill [38] to solve steady-state scalar linear hyperbolic equation but did not attract people's attention. A major development of the DG method was carried out in a series of papers [4, 6–8, 10], where the DG spatial approximation was combined with explicit Runge-Kutta time discretization to develop Runge-Kutta DG (RKDG) methods and a general framework of DG methods was estab-