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Hermite Type Spline Spaces over Rectangular Meshes with Complex Topological Structures

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Abstract. Motivated by the magneto hydrodynamic (MHD) simulation for Tokamaks with Isogeometric analysis, we present splines defined over a rectangular mesh with a complex topological structure, i.e., with extraordinary vertices. These splines are piecewise polynomial functions of bi-degree (d,d) and $\mathcal{C}^r$ parameter continuity. And we compute their dimension and exhibit basis functions called Hermite bases for bicubic spline spaces. We investigate their potential applications for solving partial differential equations (PDEs) over a physical domain in the framework of Isogeometric analysis. For instance, we analyze the property of approximation of these spline spaces for the L^2-norm; we show that the optimal approximation order and numerical convergence rates are reached by setting a proper parameterization, although the fact that the basis functions are singular at extraordinary vertices.

AMS subject classifications: 65D05, 65U07, 65U30

Key words: Spline, meshes with complex topological structures, extraordinary vertices, dimension and basis, isogeometric analysis, MHD simulation.

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1 Introduction

The finite element method (FEM) is a powerful tool that is often used to derive accurate and robust scheme for the approximation of the solution of PDEs. We are concerned with MHD equations applied to the edge plasma of fusion devices as Tokamaks. In this context of strongly magnetized plasma, the finite element formulation faces some difficulties such as the divergence-free constraint and the high anisotropy of transport processes.

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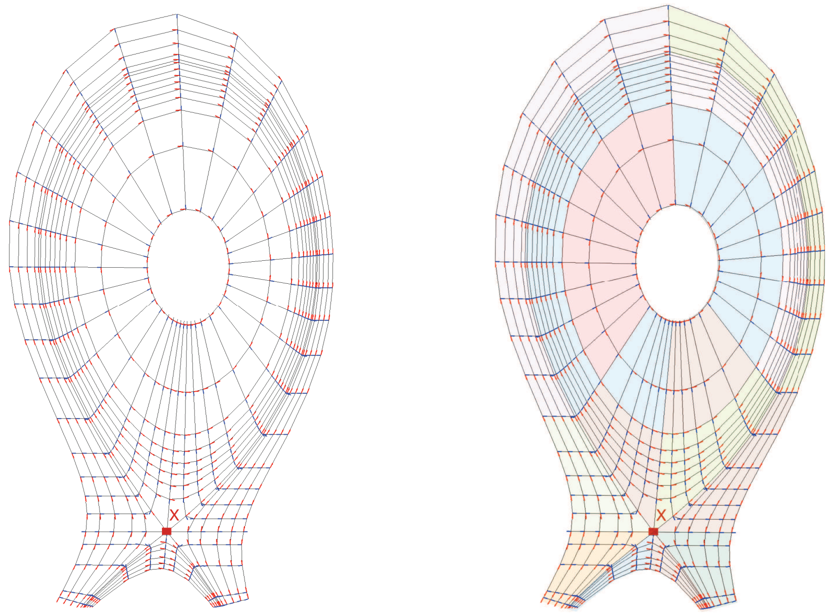

Figure 1: The structured mesh of the target application and its block structured mesh.

Higher anisotropies suggest the use of meshes aligned with the principal directions of the transport processes [1]. Quadrilateral (2D) and hexahedral (3D) meshes, called structured meshes, are the most convenient for alignment and lead to a reduction in the approximation error. In [2], isoparametric bicubic Hermite elements are used to solve the Grad-Shafranov equation, the equilibrium in the resistive MHD model, over a physical domain by aligning with concentric-circle-like principal directions of transport processes. They introduced a polar-coordinate-like transformation to construct a global coordinate system to achieve the desirable properties of the classical cubic Hermite element [3]. However, to align the principal directions of the transport processes in our target application for high-confinement Tokamaks, a structured mesh is involved, as shown on the left side of Fig. 1. Different from a regular structured mesh, there is an extraordinary vertex (the X point in Fig. 1). In this paper, we present splines defined on a rectangular mesh with a complex topological structure, i.e. the mesh allows extraordinary vertices. To solve PDEs, the properties of approximation and numerical convergence rates of these splines are discussed.

NURBS, tensor product B-splines [4], hierarchical B-splines [5–7], LR-splines [8] and T-splines [9] are often used as shape functions to generate a parameterization. Their meshes are aligned with these directions by this parameterization. However, without an extraordinary vertex, their meshes are associated with simpler topological structures than that needed for the target application. In other words, the structured mesh on the left side of Fig. 1 has to be decomposed when these splines are treated as shape functions.