A Comparison of Higher-Order Weak Numerical Schemes for Stopped Stochastic Differential Equations

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Abstract. We review, implement, and compare numerical integration schemes for spatially bounded diffusions stopped at the boundary which possess a convergence rate of the discretization error with respect to the timestep h higher than $\mathcal{O}(\sqrt{h})$. We address specific implementation issues of the most general-purpose of such schemes. They have been coded into a single Matlab program and compared, according to their accuracy and computational cost, on a wide range of problems in up to $\mathbb{R}^{48}$. The paper is self-contained and the code will be made freely downloadable.

1 Introduction

Producing numerical approximations, via Monte Carlo simulations, to the expected value of functionals involving stochastic differential equations (SDEs) stopped at a boundary is pervasive in scientific computing. Usually, the quantities of interest are the mean first passage time of a walker from that domain [33] or the pointwise solution of a boundary value problem (BVP) [9]. Some relevant fields where it encounters application are: biology [35], barrier problems in finance [13], mixing of fluids [18] and analysis of noisy dynamical systems [34]—where the interest often lies on first exit times—, as well as medical imaging [23], chemistry [26], design of integrated circuits [4], and high-performance supercomputing [1]—more focused on the connection to BVPs. It is well known that

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the convergence rate with respect to the timestep h of the approximation to the expected value—the weak convergence rate of the numerical scheme—is only $\mathcal{O}(\sqrt{h})$, unless specific measures to handle the interaction between the diffusion random path and the boundary are undertaken [16]. This compares very unfavourably with the case of diffusions in free space, where even the most basic method—namely Euler-Maruyama's—has a weak order $\mathcal{O}(h)$ [20]. In the last two decades, a number of weak schemes for bounded diffusions have been put forward seeking to raise the weak convergence rate to $\mathcal{O}(h^\delta)$, with $1/2 < \delta \leq 1$. Henceforth, we will refer to a scheme with $\delta > 1/2$ as a "higher-order scheme". (We will use both terms "scheme" and "integrator" interchangeably.)

In applications, such a low exponent as $1/2$ means that in order to bound the total error by a tolerance a, a very small h is needed, which leads to a cost $\mathcal{O}(a^{-4})$ [10] and so to unpractically lengthy simulations if good accuracy is sought for. This exponent stems from the need to balance the statistical (Monte Carlo) error with the weak order of convergence of the integrator (i.e. the bias). In fact, the exponent can be split as $2+1/\delta$, where 2 and $1/\delta$ are the contribution of either factor. Consequently, raising δ to 1 means that the cost drops to $\mathcal{O}(a^{-3})$. In order to reduce it further, a suitable variance reduction method (tackling the '2' in the exponent) and/or a $\delta > 1$ are required.

Giles' Multilevel method [10] is the algorithm of choice for variance reduction, and has truly revolutionized the field of stochastic simulation. In it, several versions of the same stochastic paths—solved at ever smaller values of h—are combined into a telescopic estimator of the expected value in such a way that the rougher versions serve as control variates for the finer ones, thus reducing the overall variance. Importantly, within the Multilevel loop the trajectories are numerically integrated with a legacy scheme such as those considered in this article. The Multilevel method was first extended to stopped diffusions in [17] using the Euler-Maruyama scheme as integrator, achieving a simulation cost of $\mathcal{O}(a^{-3}|\log a|^{1/2})$. This has been recently reduced down to $\mathcal{O}(a^{-2}|\log a|^3)$ in [11], by improving the convergence rate of the variance to linear and replacing the Euler-Maruyama integrator by the Gobet-Menozzi one. The difficulty of combining Multilevel with a given integrator lies in the analysis of the variance, which depends on the *strong* convergence properties of the integrator. In any case, the *weak* order of convergence of the Multilevel estimator is that of the integrator inside it. Moreover, the Multilevel error estimate (see [10]) also relies on that weak order being known in advance—as it is the case whenever the SDE is to be solved to within a given tolerance a.

For unbounded diffusions, there exist superlinear weak integrators [20]. Moreover, Talay-Tubaro extrapolation can be used [36], also combined with Multilevel [10, 21]. On the other hand, while there are no superlinear integrators for bounded SDEs (to the authors' best knowledge), extrapolation could in principle still be used [29, chapter 5]. As a related idea, linear regression was used in [24] to substantially reduce the bias of bounded SDEs. But again, all those strategies critically depend on the prior knowledge of the integrator's order δ.

Given the importance of this topic, it may be surprising that no systematic comparison of such higher-order schemes has been carried out before (as far as we know). This