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Boundary Control Problems in Convective Heat Transfer with Lifting Function Approach and Multigrid Vanka-Type Solvers

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Abstract. This paper deals with boundary optimal control problems for the heat and Navier-Stokes equations and addresses the issue of defining controls in function spaces which are naturally associated to the volume variables by trace restriction. For this reason we reformulate the boundary optimal control problem into a distributed problem through a lifting function approach. The stronger regularity requirements which are imposed by standard boundary control approaches can then be avoided. Furthermore, we propose a new numerical strategy that allows to solve the coupled optimality system in a robust way for a large number of unknowns. The optimality system resulting from a finite element discretization is solved by a local multigrid algorithm with domain decomposition Vanka-type smoothers. The purpose of these smoothers is to solve the optimality system implicitly over subdomains with a small number of degrees of freedom, in order to achieve robustness with respect to the regularization parameters in the cost functional. We present the results of some test cases where temperature is the observed quantity and the control quantity corresponds to the boundary values of the fluid temperature in a portion of the boundary. The control region for the observed quantity is a part of the domain where it is interesting to match a desired temperature value.

AMS subject classifications: 49J20, 65N55

Key words: Boundary optimal control, multigrid and domain decomposition algorithms, heat and Navier-Stokes equations.

1 Introduction

The optimal control of the heat and Navier-Stokes equations is an interesting subject due to a wide range of applications. In particular, coupled thermo-fluid mechanics optimal

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control problems are very challenging due to their difficulties both in the theoretical formulation and in the construction of the computational algorithm. The implementation of a computationally feasible and robust boundary control algorithm is a nontrivial task. The first nontrivial issue is the definition and numerical implementation of the boundary control in a large functional space which is naturally associated to the volume variables. A control boundary constraint or a cumbersome numerical implementation can seriously limit the set of possible solutions. For this reason in many circumstances it is desirable to transform a boundary optimal control problem into a distributed problem through the lifting function approach of nonhomogeneous boundary conditions. With the lifting function approach, boundary controls are defined in the appropriate function boundary spaces restriction of the natural state volume spaces. Another challenging task in the study of optimal control problems is the numerical solution of the optimality system. The coupled implicit solution of the state, adjoint and control equations may allow a stronger and more robust solution process, as the oscillations that would be induced in the case of uncoupled solution become negligible. The uncoupled solution of the state and the adjoint system brings to limits in the choice of the penalty parameter for the tracking term in the objective functional. In fact in the limit of this parameter tending to infinity the solution tends to lose smoothness and the uncoupled algorithm cannot converge. For a strong and robust optimization the optimality system must be solved in a coupled way. To this end, we propose the use of multigrid algorithms with domain decomposition Vanka-type smoothers. With these the problem is split into small blocks of finite element subdomains with a small number of degrees of freedom, so that an optimal solution is computed by solving the fully coupled state-adjoint system on each subdomain. In this way large penalty parameters for the tracking term can be used and a better matching with the desired state can be attained.

Various works have been presented in literature for this class of problems, see for example [13–15] and references therein. The analysis of a two-dimensional problem for the Boussinesq equations is treated in [15]. Neumann boundary optimal control problems for the stationary Boussinesq equations including solid media are considered in [13]. In [14], a linear feedback control problem for the Boussinesq equations is studied. In this paper, we consider a similar optimal control problem for the heat and Navier-Stokes equations in which velocity influences temperature but the effects of temperature on the velocity and pressure fields, such as buoyancy, are neglected. This type of one-way coupling occurs in the case where forced convection is the dominant physical mechanism for heat transfer. The control is performed through the boundary conditions of temperature on all or part of the boundary. These boundary controls are implemented in the form of lifting functions, since this method brings several advantages from theoretical and computational points of view [3,17].

The implementation of boundary controls is a nontrivial task, especially when one wants to search for controls in natural trace spaces. In [18,19] the trace space $H^{1/2}(\Gamma)$ is used for the control, but this involves the introduction of a norm on this space that is realized by means of a stabilized hypersingular boundary integral operator. This con-