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On the Wall Shear Stress Gradient in Fluid Dynamics

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Abstract. The gradient of the fluid stresses exerted on curved boundaries, conventionally computed in terms of directional derivatives of a tensor, is here analyzed by using the notion of intrinsic derivative which represents the geometrically appropriate tool for measuring tensor variations projected on curved surfaces. Relevant differences in the two approaches are found by using the classical Stokes analytical solution for the slow motion of a fluid over a fixed sphere and a numerically generated three dimensional dynamical scenario. Implications for theoretical fluid dynamics and for applied sciences are finally discussed.

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1 Introduction

Fluids interact with various types of boundaries whose shape can be geometrically simple but also extremely complicated and irregular as it happens, for instance, in geophysical or biophysical situations. A fluid flow, apart for extreme astrophysical situations, is a phenomenon occurring in flat Euclidean space. On the other hand, the boundaries interacting with the flow are commonly curved surfaces embedded on flat space.

In some fluid dynamical contexts it is important to evaluate the changes of specific physical quantities on these surfaces, leading to the necessity to adopt specific methods for handling the variations of tensors on curved manifolds as described by Differential

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Geometry. The use of these mathematical tools is central in order to define and to extract specific physical quantities both from experiments and numerical simulations.

In many situations, the stresses exerted on the wall by the flow have great importance in order to predict possible wall failures. It appears natural then to evaluate the wall shear stress (WSS), defined as the force per unit area exerted by the fluid tangentially to the wall [10, 23], although also its time and space variations, the so called Wall Shear Stress Gradient (WSSG), have relevance. For instance, the study of WSS and WSSG represents a key element for the investigations of the development of life-threatening vascular pathologies as atherosclerosis, aneurysms and gas embolism [10,13,15,16,25,34]. In the medical Literature, experiments and numerical analyses suggest that abnormal values of WSSG can induce a vessel remodelling, influencing the shape of the endothelial cells and promoting inflammatory responses. Endothelium (i.e. the innermost arterial layer in direct contact with blood) is in fact equipped with numerous mechanoreceptors sensitive both to WSS and its spatial variations which may induce cell injury and weakening [10,13,15,16,27,35]. In particular, abnormal hemodynamic conditions may increase the intracellular permeability, promote atherogenic signalling pathways, and thus induce the progressive intimal thickening involved in the atherosclerosis [6,7,13,36]. Experimental and numerical estimates have also shown that both WSS and WSSG may play a crucial role in the weakening of the arterial walls leading to the progressive dilation of a blood vessel and consequently to the onset of aneurysms [17, 25, 32]. Moreover the WSSG is often applied in the optimization of the biomedical devices design such as endovascular grafts and stents [11, 20, 26, 31, 33].

The aim of this work is to present the appropriate definition of the WSSG by using mathematical tools commonly adopted in General Relativity, Quantum Field Theory, Condensed Matter Physics and Continuum Mechanics.

In the Literature, in fact, the WSSG has been historically defined through the directional derivatives of the WSS. However by using the *vielbein* formalism (known equivalently as the tetrad, rigid frames, anholonomic bases, or Cartan frames theory), one can show that the correct definition must involve another tool known as intrinsic derivative. We find relevant differences between the two definitions of WSSG by using the classical Stokes analytical solution for the slow motion of a fluid over a fixed sphere and a CFD three dimensional model of aortic aneurysm. These results have importance both for experiments and numerical simulations.

The article is organized as follows: Section 2 introduces the differential geometry tools necessary to provide a correct definition of WSSG whereas the results obtained both for the analytical and numerical estimates are described in Section 3. Finally the implications of our results in comparison to the existing Literature are discussed in Section 4.

2 Theoretical framework

We start our analysis from the general equations of fluid dynamics, i.e. the equation of continuity and the equations of motion. The energy equation is here neglected for the