Commun. Comput. Phys.
doi: 10.4208/cicp.150414.101014a

Vol. **17**, No. 3, pp. 721-760
March 2015

Discontinuous-Galerkin Discretization of a New Class of Green-Naghdi Equations

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Received 15 April 2014; Accepted (in revised version) 10 October 2014

Abstract. We describe in this work a discontinuous-Galerkin Finite-Element method to approximate the solutions of a new family of 1d Green-Naghdi models. These new models are shown to be more computationally efficient, while being asymptotically equivalent to the initial formulation with regard to the shallowness parameter. Using the free surface instead of the water height as a conservative variable, the models are recasted under a *pre-balanced* formulation and discretized using a nodal expansion basis. Independently from the polynomial degree in the approximation space, the preservation of the motionless steady-states is automatically ensured, and the water height positivity is enforced. A simple numerical procedure devoted to stabilize the computations in the vicinity of broken waves is also described. The validity of the resulting model is assessed through extensive numerical validations.

AMS subject classifications: 76M10, 65M10

Key words: Green-Naghdi equations, discontinuous Galerkin, shallow water equations, wave breaking, nonlinear and dispersive water waves.

1 Introduction

Depth-averaged equations are widely used in coastal engineering for the simulation of nonlinear waves propagation and transformations in nearshore areas. The full description of surface water waves in an incompressible, homogeneous, inviscid fluid, is provided by the free surface Euler (or water waves) equations but this problem remains mathematically and numerically challenging. As a consequence, the use of depth averaged equations helps to reduce the three-dimensional problem to a two-dimensional problem, while keeping a good level of accuracy in many configurations.

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Many Boussinesq-like models are used nowadays and a detailed review can be found in [44] and the recent monograph [43]. Denoting by L the typical horizontal scale of the flow and h_0 the typical depth, the shallow water regime usually corresponds to the configuration where $\mu := \frac{h_0^2}{L^2} \ll 1$. If approximations of order $\mathcal{O}(\mu^2)$ of the free surface Euler equations are furnished by the Boussinesq-type (BT equations in the following) equations, see [54, 56, 60] for instance, an additional smallness amplitude assumption on the typical wave amplitude a is classically performed: $\varepsilon := \frac{a}{h_0} = \mathcal{O}(\mu)$. This assumption often appears as too restrictive for many applications in coastal oceanography. Removing the small amplitude assumption while still keeping all the $\mathcal{O}(\mu)$ terms, we obtain the so-called *Green-Naghdi* equations (GN equations in the following) [36], also referred to as *Serre* equations [66] or *fully non-linear Boussinesq* equations [80].

A large number of numerical methods have been developed in the past few years for the BT equations. Let us mention for instance some Finite-Difference (FDM in the following) approaches [51, 56, 69, 79], Finite-Element methods (FEM in the following) [48,62,70,77], Finite-Volume discretizations (FVM in the following) for 1d equations [23], hybrid FDM/FVM [26,27,57,68,73], or even a purely 2d FVM discretization on unstructured meshes [39], allowing for mesh refinement and flexibility for large scale simulations.

As far as flexibility is concerned, the use of discontinuous-Galerkin methods (dG methods in the following) would appear as a natural choice. Indeed, this class of method provides several appealing features, like compact discretization stencils and hp-adaptivity, flexibility with a natural handling of unstructured meshes, easy parallel computation and local conservation properties in the approximation of conservation laws. A general review of dG methods for convection dominated problems is performed in [16]. Concerning the approximation of more general problems, involving higher-order derivatives, several methods and important developments have been proposed in recent years, following [5] on Navier-Stokes equations and [17] on convection-diffusion systems. A recent review is performed in [84] and a unified analysis can be found in [4], and [28, 29], respectively for elliptic problems and both 1^{st} and 2^{nd} order problems in the framework of Friedrichs' systems.

The application of dG methods to the *Saint-Venant* equations (also called *Nonlinear Shallow Water* equations, NSW in the following) has recently lead to several improvements, see for instance [3, 30, 82, 83] and the recent review [21]. However, dG methods for BT equations have been under-investigated. In [31], a *hp*/spectral element model is introduced for the 1d enhanced equations of Nwogu [56], while the 2d equations of Peregrine [59] are studied in [32], in the flat bottom case, relying on a scalar reformulation that allowed some computational savings. This formulation is further investigated in [33], accounting for variable depth, and in [34] with the study of the enhanced equations of Madsen and Sorensen [53]. In [24, 25], an arbitrary order nodal dG-FEM is developed for the set of highly-dispersive BT equations introduced in [52], respectively in 1d and 2d on unstructured meshes. These equations have a larger range of validity and can the-