A Domain Decomposition Based Spectral Collocation Method for Lane-Emden Equations

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Abstract. A domain decomposition based spectral collocation method is proposed for numerically solving Lane-Emden equations, which are frequently encountered in mathematical physics and astrophysics. Compared with the existing methods, this method requires less computational cost and is particularly suitable for long-term computation. The related error estimates are also established, indicating the spectral accuracy of the method. The numerical performance and efficiency of the method are illustrated by several numerical experiments.

1 Introduction

The Lane-Emden equation is defined in the following form:

$$
\begin{cases}
u''(t) + \dfrac{a}{t} u'(t) + f(u(t)) = 0, & t > 0, \ a \in \mathbb{N}^+, \\
u(0) = u_0, \quad u'(0) = 0,
\end{cases}
\tag{1.1}
$$

where u_0 is the initial data and f is a continuous function in $\mathbb{R}$. Such kind of equations are frequently encountered in mathematical physics and astrophysics, including the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and the theory of thermionic currents. We refer the reader to the references [4, 11, 17] for details about the physical background of the equation.

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Due to the importance of Lane-Emden equations, there have developed a variety of numerical methods for solving them in the past decade (see [12,14,22] and the references therein). The typical and effective approach is the spectral method. Concretely speaking, Parand et al. proposed in [14] a Hermite function collocation method, Doha et al. proposed respectively in [5,6] a Jacobi rational Gauss collocation/pseudospectral method, and Gürbü and Sezer proposed in [7] a Laguerre collocation method, to solve the Lane-Emden type equations. The other recent methods include the homotopy perturbation method (cf. [13]), the variational iteration method (cf. [23]), the Adomian decomposition method (cf. [16]) and the wavelet method (cf. [18,26]).

Compared with the other methods, the spectral method has a significant advantage, that means, it can produce very accurate numerical solution (cf. [19]). However, all the spectral methods mentioned above only study the global approximation in the whole solution domain $[0,\infty)$. Hence, they are not suitable for long-term computation since we can not choose the polynomial degree N very large in actual computation (cf. [24]). The other point to be emphasized is that no error analysis has been developed for these methods. In this article, we aim to devise a domain decomposition based spectral collocation method for solving the problem (1.1), which is very suitable for long-term computation. Then, we will establish error estimates for the method and show it has spectral accuracy if the exact solution is smooth enough. Moreover, we will provide a series of numerical experiments to show the numerical performance of our method proposed.

Since the idea of the proposed domain decomposition spectral collocation method is very natural and intuitive, we'd like to briefly introduce it in this introduction section. The details of the method can be found in Section 3. To this end, we first divide the unbounded domain $[0,\infty)$ into the two subdomains $\Gamma_1 := [0,T_0]$ and $\Gamma_2 := [T_0,\infty)$, where T_0 is a given constant as desired. To tackle the singularity of the coefficient function a/t, we reformulate (1.1) as the initial value problem of an equivalent Volterra integro-differential equation:

$$
\begin{cases}
u'(t) + \displaystyle\int_0^t \frac{s^a}{t^a} f(u(s))ds = 0, & t \in \Gamma_1, \\
u(0) = u_0,
\end{cases}
\tag{1.2}
$$

together with the initial value problem of a second order differential equation

$$
\begin{cases}
u''(t) + \dfrac{a}{t}u'(t) + f(u(t)) = 0, & t \in \Gamma_2, \\
u(T_0) = u_(T_0-), \quad u'(T_0) = u'(T_0-),
\end{cases}
\tag{1.3}
$$

where $u(T_0-)$ and $u'(T_0-)$ are determined from problem (1.2). Then, the problem (1.2) is approximated by a multi-step Legendre-Gauss-Radau (LGR) collocation method based on the formulation (1.2) (cf. [2,20]). Since the previous method is very time-consuming, we then turn to use the standard multi-step LGR collocation method for solving the problem (1.1) after $t \geq T_0$, i.e. the problem (1.3) (cf. [10]). We remark that, compared with the