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An Efficient Implementation of the Divergence Free Constraint in a Discontinuous Galerkin Method for Magnetohydrodynamics on Unstructured Meshes

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Abstract. In this paper we consider a discontinuous Galerkin discretization of the ideal magnetohydrodynamics (MHD) equations on unstructured meshes, and the divergence free constraint ($\nabla\cdot\boldsymbol{B}=0$) of its magnetic field $\boldsymbol{B}$. We first present two approaches for maintaining the divergence free constraint, namely the approach of a locally divergence free projection inspired by locally divergence free elements [19], and another approach of the divergence cleaning technique given by Dedner et al. [15]. By combining these two approaches we obtain a efficient method at the almost same numerical cost. Finally, numerical experiments are performed to show the capacity and efficiency of the scheme.

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Key words: Ideal magnetohydrodynamics equations, discontinuous Galerkin method, divergence free constraint, locally divergence free projection, divergence free cleaning technique.

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1 Introduction

Many physical problems arising in a modeling process can be described by the magnetohydrodynamic (MHD) equations which model the dynamics of electrically conducting fluids (e.g. plasma). At high temperature due to ionization, all gases will change to plasma. Therefore the MHD equations are important in many physical applications. Since the equations are highly nonlinear, analytic solutions are not available for the problem. We will focus on the numerical solutions of the ideal MHD equations, represented by an hyperbolic conservation law. Also, an additional involution constraint $\nabla\cdot\boldsymbol{B}=0$ for

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its magnetic field $\boldsymbol{B}$ is needed. Besides the numerical challenges when solving such a nonlinear system, this constraint introduces additional difficulties. On the analytic level the involution constraint is always fulfilled, but numerical experiments indicate that negligence in dealing with the divergence constraint may lead to numerical instability and nonphysical solutions.

Many numerical approaches have been developed to solve conservation laws, e.g. finite volume method (FVM) and finite element method (FEM). Each of them has its advantages and disadvantages. We will focus on the discontinuous Galerkin (DG) method, which combines the flexibility of FEM with the numerical fluxes from FVM. The DG method uses piecewise basis functions which are discontinuous on the boundary of the elements. Normally they are chosen to be piecewise polynomials. Due to the discontinuity of the basis function across cell boundaries, the scheme is very flexible compared to standard continuous finite element method, such as its ability to deal with arbitrary unstructured grids with hanging nodes. Additionally, each cell can have its own polynomial degree independent of its neighbors. Furthermore, the DG scheme admits extremely local data structure (elements only communicate with its immediate neighbors) which leads to high parallel efficiency.

The first discontinuous Galerkin method was introduced by Reed and Hill [28] in 1973. They were interested in the neutron transport problem, i.e. a time independent linear hyperbolic equation. In a series of papers [9–13] Cockburn and Shu developed a framework to solve nonlinear time dependent problems, like Euler or MHD equations. For time integration they are using explicit, nonlinearly stable high order Runge-Kutta time discretizations [30] and for the spatial DG discretization they apply exact or approximate Riemann solvers as interface numerical fluxes. To avoid numerical ascillations near shocks they suggested to apply total variation bounded nonlinear limiters [29]. Due to its good properties, such as high order accuracy and parallel efficiency, the discontinuous Galerkin method has found rapid applications in diverse areas as aeroacoustics, electro-magnetism, gas dynamics and many more. Several numerical results establish the good convergence behavior and reveal an excellent level of details in numerical runs, see e.g. [1].

With regards to the numerical influence of the divergence constraint, several modifications and ideas have been developed to satisfy the constraint or at least reduce the negative impact on the numerical solution. One of the first persons to notice the impact of nonzero divergence to the stability of the numerical schemes were Brackbill and Barnes. In [5], they proposed a global projection method to stabilize their scheme. The projection needs to solve a global elliptic partial differential equation at each time step.

Another approach is given by Powell [25, 27]. The derivation of one-dimensional fluxes (for a finite volume scheme) is based on the symmetrizable form of the MHD equations. In order to symmetrize MHD, we have to add source terms proportional to $\nabla\cdot\boldsymbol{B}$, see [26]. It was discovered later that the robustness of a MHD code can be improved by adding these so called Powell-source term, see [34]. In 2002 Dedner et al. [15] introduced their hyperbolic divergence cleaning technique which has several advantages