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An Online Generalized Multiscale Discontinuous Galerkin Method (GMsDGM) for Flows in Heterogeneous Media

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Abstract. Offline computation is an essential component in most multiscale model reduction techniques. However, there are multiscale problems in which offline procedure is insufficient to give accurate representations of solutions, due to the fact that offline computations are typically performed locally and global information is missing in these offline information. To tackle this difficulty, we develop an online local adaptivity technique for local multiscale model reduction problems. We design new online basis functions within Discontinuous Galerkin method based on local residuals and some optimally estimates. The resulting basis functions are able to capture the solution efficiently and accurately, and are added to the approximation iteratively. Moreover, we show that the iterative procedure is convergent with a rate independent of physical scales if the initial space is chosen carefully. Our analysis also gives a guideline on how to choose the initial space. We present some numerical examples to show the performance of the proposed method.

AMS subject classifications: 65N12, 65N30

Key words: Multiscale method, discontinuous Galerkin method, online basis functions, heterogeneous media.

1 Introduction

In this paper, we develop an online local adaptivity technique for a class of multiscale model reduction problems. Many realistic applications involve solving problems that contain multiple scales and high contrast. Direct solution methods for these problems

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require fine-grid discretizations and result in large discrete systems that are computationally intractable. Common model reduction techniques perform the discretization of the problems on a coarse grid, which is much larger than the scales under consideration, with the aim of getting more efficient solution strategies. There are a variety of multiscale model reduction techniques based on numerical upscaling (e.g., [15,28,39]) or multiscale methods (e.g., [3,6–8,10,12,16,22–24,29]). Most of the existing techniques are based on the so called offline construction. In particular, reduced models are computed in a pre-processing step, called offline stage, before the actual simulations, called online stage, are performed. For instances, some effective media are pre-computed for methods based on numerical upscaling and some multiscale basis functions are pre-computed for multiscale finite element methods. While these methods are effective in a wide variety of applications, there are still situations for which these methods are inadequate to give reliable solutions unless a large dimensional offline space is employed. Some of these situations involve external source effects and distant effects, which are ignored by most multiscale model reduction methods since they are typically based on local constructions. Therefore, it is evident that offline procedures are sometimes not enough to give efficient reduced models. Hence, it is the purpose of this paper to design a novel multiscale model reduction method. Our proposed method is based on a combination of offline technique and an online enrichment technique. The online technique is able to produce a reduced model taking care of external sources and distant effects, without using global models. The online construction is also performed locally and adaptively in regions with more heterogeneities, giving very efficient reduced models.

Our proposed method follows the overall idea of the Generalized Multiscale Finite Element Method (GMsFEM), which is introduced in [18] and is a generalization of the classical multiscale finite element method [31] in the way that the coarse spaces are systematically enriched, taking into account small scale information and complex input spaces. Instead of conforming finite element spaces as in [18, 31], we will use in this paper discontinuous Galerkin finite element spaces, which have some essential advantages (see [10, 19]) in multiscale simulations because it allows coupling discontinuous basis functions. The discretization starts with a coarse grid and a space of snapshot functions, which are defined on coarse elements. A space reduction is then performed to obtain a much smaller offline space by means of spectral decomposition. The spectral decomposition is performed locally on coarse elements, thus the functions in the offline space are in general discontinuous across coarse edges. The offline space is used as the approximation space for the interior penalty discontinuous Galerkin (IPDG) discretization on the coarse grid for the problem under consideration, giving our generalized multiscale discontinuous Galerkin method (GMsDGM). We remark that the offline space is computed only once in the pre-processing offline stage, and the same set of basis functions is used for any given source terms and boundary conditions. A-priori error estimate can be derived as in [10, 21, 22] showing that the error is inverse proportional to the first eigenvalue corresponding to the first eigenfunction that is not used in the construction of the reduced space. Since the aim of the paper is the new online locally adaptive procedure