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Fourth Order Exponential Time Differencing Method with Local Discontinuous Galerkin Approximation for Coupled Nonlinear Schrödinger Equations

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Abstract. This paper studies a local discontinuous Galerkin method combined with fourth order exponential time differencing Runge-Kutta time discretization and a fourth order conservative method for solving the nonlinear Schrödinger equations. Based on different choices of numerical fluxes, we propose both energy-conserving and energy-dissipative local discontinuous Galerkin methods, and have proven the error estimates for the semi-discrete methods applied to linear Schrödinger equation. The numerical methods are proven to be highly efficient and stable for long-range soliton computations. Extensive numerical examples are provided to illustrate the accuracy, efficiency and reliability of the proposed methods.

AMS subject classifications: 65M20, 65M60, 65Z05

Key words: Exponential time differencing, local discontinuous Galerkin, nonlinear Schrödinger equation, energy conserving, error estimate.

1 Introduction

The N-coupled nonlinear Schrödinger equation (NLSE) is widely used to model a number of important physical phenomena, including propagations of solitary waves in optical fibers [2], deep water turbulence [25] and laser beams [31]. In 1967, the 2-coupled NLSE was first derived in [3] to study two interacting nonlinear packets in a dispersive

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and conservative system. The N-coupled NLSE was proposed in [28] to accurately evaluate the signal distortion in optical communication fibers, and further studied in many literatures. In one space dimension, it takes the form of

$$iu_{nt}+i\alpha_n u_{nx}+\epsilon_n u_{nxx}+\Big(\sum_{m=1}^{N}\beta_{nm}|u_m|^2\Big)u_n=0, \quad n=1,2,\cdots,N, \tag{1.1}$$

where u_n represent the amplitude of the pulse envelops, α_n indicate the group velocities for polarization components, ϵ_n are the group velocity dispersion parameters, and β_{mn} are nonlinearity parameters responsible for the Self Phase Modulation [1].

A vast amount of literature can be found on the numerical approximation of the NLSEs. Many different types of numerical methods, including finite difference, finite element, finite volume, and spectral methods, have been designed, see for example, [7, 15, 18, 20, 21, 29, 30, 32, 34, 36] and references therein. The analysis of some finite difference methods can be found in [7, 20, 34]. Recently, [33] studies a finite difference scheme for 2-coupled NLSEs, and they have shown the boundedness of the numerical solution in the discrete L_∞ norm. The numerical solutions of the NLSEs by both finite element Galerkin and finite difference methods are investigated in [15]. It appears that the Galerkin method produces more acceptable results for a wider range of parameters. Many finite element methods have been studied for the NLSEs, see [21,29] and the references therein.

However, most articles mentioned above do not consider the group velocity for polarization components, except for [20,36]. In this paper, we consider (1.1) with $\alpha_n \neq 0$, since α_n has significant meanings in nonlinear fiber optics. According to [1], even a single-mode fiber can support two degenerate modes that are polarized in two orthogonal directions. Especially, in high-birefringence fibers, the group velocity mismatch between the fast and slow components of the input pulse cannot be neglected, which means the polarization components α_n in (1.1) cannot be ignored.

The numerical methods discussed here are the discontinuous Galerkin (DG) methods. They belong to a class of finite element methods using piecewise polynomial spaces for both the numerical solution and the test functions. They were originally devised to solve hyperbolic conservation laws with only first order spatial derivatives, e.g. [11,13]. They have many attractive advantages, including the allowance of arbitrarily unstructured meshes, a compact stencil and easy h-p adaptivity. The DG methods were later generalized to the local discontinuous Galerkin (LDG) methods by Cockburn and Shu to solve the convection-diffusion equation [12]. As a result, the LDG methods have been successfully applied to solve various partial differential equations (PDEs) containing higher-order derivatives. For the single and 2-coupled NLSEs, a Runge-Kutta LDG method was first developed in [36], in which they provided the stability analysis and a $k+1/2$-th error estimate for the linearized problem.

The performance of several fourth order temporal discretizations for the single NLSEs is compared in [23]. Methods considered there include the exponential time-differencing