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Mathematical Principles of Anisotropic Mesh Adaptation[†]

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Abstract. Mesh adaptation is studied from the mesh control point of view. Two principles, equidistribution and alignment, are obtained and found to be necessary and sufficient for a complete control of the size, shape, and orientation of mesh elements. A key component in these principles is the monitor function, a symmetric and positive definite matrix used for specifying the mesh information. A monitor function is defined based on interpolation error in a way with which an error bound is minimized on a mesh satisfying the equidistribution and alignment conditions. Algorithms for generating meshes satisfying the conditions are developed and two-dimensional numerical results are presented.

Key words: Mesh adaptation; anisotropic mesh; equidistribution; alignment; error analysis; finite element.

1 Introduction

Many partial differential equations (PDEs) arising from science and engineering have a common feature that they have a small portion of the physical domain where small node separations are required to resolve large solution variations. Examples include problems having boundary layers, shock waves, ignition fronts, and sharp interfaces in fluid dynamics, the combustion and heat transfer theory, and groundwater hydrodynamics. Numerical solution of these PDEs using a uniform mesh can be formidable when the systems involve more than two spatial dimensions since the number of mesh nodes required may become large. To improve efficiency and accuracy of numerical solution it is natural to put more mesh nodes in regions of large solution variation than the rest of the physical domain.

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With this basic idea of mesh adaptation, the number of mesh nodes required can be much smaller; thus significant economies are gained.

Essential to mesh adaptation is the ability to control the *size*, *shape*, and *orientation* of mesh elements throughout the domain. Traditionally, research has been concentrated on isotropic mesh adaptation where mesh elements are adjusted only in size according to an error estimate or indicator while their shape is kept close to being equilateral; e.g., see books [2,15,27,52] and references therein. However, isotropic meshes often tend to use too many elements in regions of large solution error. This is especially true when problems have an anisotropic feature that the solution changes more significantly in one direction than the others. Full benefits of mesh adaptation can only be taken by simultaneously adjusting the size, shape, and orientation of mesh elements according to the behavior of the physical solution. This often results in an anisotropic mesh, a mesh having elements of large aspect ratio.

The well-known equidistribution principle [11, 20] has been playing an important role in mesh adaptation. It entails finding a mesh which evenly distributes an error density among the mesh elements. The principle has been serving as a guideline in developing mesh adaptation strategies, and most existing adaptive mesh algorithms are more or less related to it. Unfortunately, it is known [49] that the equidistribution principle is insufficient to determine an anisotropic mesh in multi-dimensions. Great effort has been made in the last decade to develop multi-dimensional generalizations of the equidistribution principle and/or other principles for anisotropic mesh adaptation; e.g., see [4, 19, 32, 37, 41, 42, 50]. Given a physical domain $\Omega \subset \Re^n$ $(n \geq 1)$, an adaptive mesh thereon can be generated as the image of a logical or computational mesh under a coordinate transformation $x = x(\xi) : \Omega_c \to \Omega$, where Ω_c is the computational domain artificially chosen for the purpose of mesh generation. Denote by $\boldsymbol{J} = (\partial x)/(\partial \xi)$ the Jacobian matrix of the coordinate transformation and $J = \det(\boldsymbol{J})$ its determinant. Motivated by a discrete constrained optimization problem, Steinberg and Roache [50] define $x = x(\xi)$ by minimizing the functional $\int_{\Omega_c} J^2 d\xi$ subject to the global implicit constraint $\int_{\Omega_c} J d\xi = |\Omega|$, with intention to keep element volume constant. These ideas of relating mesh adaptation functionals to equidistribution and using global implicit constraints are studied extensively by Knupp and Robidoux [42]. Upon studying linear interpolation error on triangular elements, D'Azevedo and Simpson [19] suggest that the coordinate transformation be chosen to minimize the gradient of interpolation error and thus to satisfy

$$\boldsymbol{J}^T H(v)^T H(v) \boldsymbol{J} = cI, \qquad \text{in } \Omega_c \tag{1.1}$$

where c is a constant and $H(v)$ denotes the Hessian of a function v. Huang and Sloan [37] choose the coordinate transformation such that the function, when transformed into the new coordinate, has the same change rate at every point and in every direction. This results in

$$\boldsymbol{J}^T \left(I + \nabla v \nabla v^T \right) \boldsymbol{J} = cI \qquad \text{in } \Omega_c \tag{1.2}$$

for some constant c. Huang [32] generalizes the ideas of [19, 37] to the case with an arbitrary $n \times n$ symmetric and positive definite matrix $M = M(x)$ (named *a monitor*