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A Posteriori Error Estimates for the Weak Galerkin Finite Element Methods on Polytopal Meshes

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Abstract. In this paper, we present a simple a posteriori error estimate for the weak Galerkin (WG) finite element method for a model second order elliptic equation. This residual type estimator can be applied to general meshes such as hybrid, polytopal and those with hanging nodes. We prove the reliability and efficiency of the estimator. Extensive numerical tests demonstrate the effectiveness and flexibility of the mesh refinement guided by this error estimator.

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Key words: Weak Galerkin, finite element methods, second-order elliptic problems, a posteriori error estimate, polytopal meshes.

1 Introduction

Adaptive finite element methods [20] are widely used in modern computational science and engineering to obtain better accuracy with minimal effort. It can be achieved through adaptive mesh refinement that generates a mesh tailored in reducing computational errors at places of great need. Adaptive mesh refinement will be more local and effective for the finite element methods that allow general mesh [11, 13]. In recent years, many numerical schemes have been developed and analyzed on general polytopal mesh such as HDG method, mimetic finite difference method, virtual element method and hybrid high-order method [5, 6, 10, 22].

A posteriori error analysis enables a measure of the reliability and efficiency of a particular numerical scheme employed for approximating the solution of partial differential

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equations [1, 3, 24]. This result is a computable estimator that is an indicator of where the error is potentially large and specific elements need to be refined. A posteriori error analysis has been studied excessively for finite element methods with discontinuous approximation and we list few recent development of residual based a posteriori error estimates for the second order elliptic problems [2, 4, 7, 15, 16, 21, 29].

The weak Galerkin method is a natural extension of the standard Galerkin finite element method for discontinuous approximations. Its finite element formulation can be derived directly from the weak form of the corresponding partial differential equation where classical derivatives are substituted by weakly defined derivatives with a parameter-free stabilizer. Therefore, the weak Galerkin method has the flexibility of employing discontinuous elements and, at the same time, share the simple formulations of the continuous finite element methods. An important feature of the WG methods is allowing the use of general polytopal meshes [18,19,25,27]. The importance of such feature in adaptive finite element methods is well stated in [11, 13].

An a posteriori error estimator has been developed and analyzed for the WG method in [9], in which only simplicial elements are considered. In this paper, we establish a new simple a posteriori error estimator for the weak Galerkin finite element approximation for using in the approximation of a second order elliptic equation. This error estimator has several unique features: 1) it can be applied on a general mesh such as polygonal/polyhedral mesh, hybrid mesh and mesh with hanging node. This feature is highly desirable in adaptive mesh refinement. 2) Our error estimator is simple containing only one term, a parameter free stabilizer, in addition to data oscillation. The common terms in error estimators such as area residual and flux jumps do not appear in our a posteriori estimator. Since the stabilizer has already been calculated in the process of obtaining the WG finite element approximation, there is no additional cost to compute the error estimator other than a high order data oscillation. 3) We obtain efficiency directly due to the simplicity of the error estimator. We prove the reliability of the a posteriori error estimator. Extensive numerical examples have been studied on different polygonal meshes to demonstrate the effectiveness and flexibility of the a posteriori error analysis.

For simplicity, we consider a simple model problem that seeks an unknown function u satisfying

$$-\Delta u = f, \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a polytopal domain in $\mathbb{R}^d$ (polygonal or polyhedral domain for $d=2,3$).

2 Weak Galerkin finite element schemes

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimensions or polyhedra in three dimensions satisfying a set of conditions specified in [25].