Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0024

Vol. **26**, No. 2, pp. 506-530
August 2019

Point Integral Method for Elliptic Equations with Variable Coefficients on Point Cloud

Zhen Li[1], Zuoqiang Shi[2,*] and Jian Sun[1]

[1] *Yau Mathematical Sciences Center, Tsinghua University, Beijing, China, 100084.*
[2] *Department of Mathematical Sciences & Yau Mathematical Sciences Center, Tsinghua University, Beijing, China, 100084.*

Received 25 January 2018; Accepted (in revised version) 16 August 2018

Abstract. In this paper, we generalize the point integral method to solve the general elliptic PDEs with variable coefficients and corresponding eigenvalue problems with Neumann, Robin and Dirichlet boundary conditions on point cloud. The main idea is using integral equations to approximate the original PDEs. The integral equations are easy to discretize on the point cloud. The truncation error of the integral approximation is analyzed. Numerical examples are presented to demonstrate that PIM is an effective method to solve the elliptic PDEs with smooth coefficients on point cloud.

AMS subject classifications: 65N12, 65N25, 65N75

Key words: Point integral method, elliptic equation, variable coefficients, point clouds.

1 Introduction

In the past decades, data play more and more important role in sciences and engineering, even in our daily life. Among varieties of models in data analysis, manifold based model attracts more and more attentions [3,6–9,16,18,20,21,27–29,31,32,34–36,40]. In the manifold model, data or images are associated to a smooth manifold embedded in a high dimensional Euclidean space. PDEs on the manifold, particularly the Laplace-Beltrami equation, give us powerful tools to reveal the underlying structure of the manifold. Usually, in data analysis and image processing, the manifold is represented as a collection of unstructured points in high dimensional space, which is refereed as point cloud. To solve PDEs in point cloud, the traditional methods for PDEs on Euclidean space may not work.

Among varieties of manifolds, 2D surfaces embedded in $\mathbb{R}^3$ play very important role in many scientific and engineering problems, including material science [5,13], fluid flow

*Corresponding author. *Email addresses:* `lishen03@gmail.com` (Z. Li), `zqshi@tsinghua.edu.cn` (Z. Shi), `jsun@math.tsinghua.edu.cn` (J. Sun)

[15, 17], biology and biophysics [1, 2, 14, 33]. Many numerical methods to solve surface PDEs have been developed in the past decades, such as surface finite element method [12], level set method [4,41], grid based particle method [22,23] and closest point method [30, 37]. Finite element method has very good theoretical properties. It is shown that FEM converges quadratically in L^2 norm and linearly in H^1 norm for solving the Poisson equations [11]. However, to apply FEM, global mesh is needed which is not easy to generate especially in high dimensional space. The other methods, including level set method, grid based particle method, closest point method, also need extra information. These information is not easy to obtain from point cloud, especially in high dimensional space.

Due to the development of data and imaging science, solving PDEs on high dimensional unstructured point cloud absorb more and more attentions. Many researchers from different areas are trying to find alternative methods to discretize differential operators on high dimensional point cloud. Liang et al. proposed a method based on local least square approximations of the manifold [26]. Later, Lai et al. proposed the local mesh method to approximate differential operators on point cloud [19]. Despite of lack of proof, moving least square and local mesh based methods achieve high order accuracy and have achieved great successes in many applications.

The other numerical method in point cloud for Poisson equation is the point integral method (PIM) [25, 39]. The main idea of the point integral method is to approximate the Poisson equation via an integral equation:

$$\begin{aligned} -\int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y}) \bar{R}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mathbf{y} \approx & \frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mathbf{y} \\ & -2\int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}, \end{aligned} \tag{1.1}$$

where $\mathbf{n}$ is the out normal of $\partial\mathcal{M}$, $\mathcal{M}$ is a smooth d-dimensional manifold embedded in $\mathbb{R}^N$, $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$. $R_t(\mathbf{x},\mathbf{y})$ and $\bar{R}_t(\mathbf{x},\mathbf{y})$ are kernel functions given as follows

$$R_t(\mathbf{x},\mathbf{y}) = C_t R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right), \quad \bar{R}_t(\mathbf{x},\mathbf{y}) = C_t \bar{R}\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right),$$

where $C_t = \frac{1}{(4\pi t)^{d/2}}$ is the normalizing factor. $R \in C^2(\mathbb{R}^+)$ be a positive function that is integrable over $[0,+\infty)$,

$$\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s. \tag{1.2}$$

$\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator (LBO) on $\mathcal{M}$. The integral equation is much easier to discretize on point clouds with proper quadrature rule.

In this paper, we generalize PIM to solve general elliptic equations on manifold $\mathcal{M}$. We assume that $\mathcal{M} \in C^2$ is a compact d-dimensional manifold isometrically embedded in $\mathbb{R}^N$ with the standard Euclidean metric and $d \leq N$. If $\mathcal{M}$ has boundary, the boundary,