Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0063

Vol. **26**, No. 2, pp. 434-468
August 2019

Double Source Transfer Domain Decomposition Method For Helmholtz Problems

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Received 12 March 2018; Accepted (in revised version) 16 July 2018

Abstract. We propose and study a double source transfer domain decomposition method (Double STDDM) for solving the truncated perfectly matched layer approximation in the bounded domain of Helmholtz problems. Based on the decomposition of the domain into non-overlapping layers and instead of transferring the source along one direction in STDDM [Z. Chen and X. Xiang, 2013], Double STDDM transfers the source in each layer along two directions, which can capture of the reflection information for heterogenous media. Double STDDM is an iterative scheme, and in each iteration, it first transfers the source from down to up and produces the Up wave (the wave propagating from down to up), and then transfers the source from up to down and produces the Down wave (the wave propagating from up to down). The output of Double STDDM is the summation of the Up and Down waves that are produced during the iteration. By using the fundamental solution of the PML equation, the convergence of Double STDDM is proved for the case of a constant wavenumber.
Numerical examples are included to show the efficient performance of using Double STDDM as a preconditioner both for the problems with constant and heterogenous wavenumbers. For problems with a low velocity contrast, the number of iterations is independent of the wavenumber and mesh size, whereas for problems with a high velocity contrast, double STDDM performs much better than STDDM.

AMS subject classifications: 65F08, 65N55

Key words: Helmholtz equation, high frequency waves, perfect matched layers, source transfer domain decomposition method.

1 Introduction

We propose and study a Double STDDM (source transfer domain decomposition method) for solving the unbounded Helmholtz problem:

$$\Delta u+k^2u=f \quad \text{in} \quad \mathbb{R}^2, \tag{1.1}$$

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$$r^{1/2}\left(\frac{\partial u}{\partial r}-\mathbf{i}ku\right)\to 0 \quad \text{as} \quad r=|x|\to\infty, \tag{1.2}$$

where $k(x)=\omega/c(x)>0$ is the wavenumber and $f\in H^1(\mathbb{R}^2)'$ has compact support, where $H^1(\mathbb{R}^2)'$ is the dual space of $H^1(\mathbb{R}^2)$. We assume that no reflection waves exist that propagate from outside a bounded domain; e.g., $k(x)$ may be a constant outside a bounded domain. $c(x)$ is known as the velocity field, and ω is the frequency. We remark that the results in this paper can be easily extended to solve three-dimensional Helmholtz problems.

The Helmholtz equation (1.1)-(1.2) appears in diverse scientific and engineering applications, including acoustics, elasticity, and electromagnetics. An efficient algebraic solver for large wavenumber discrete Helmholtz equation based on finite difference or finite element discretization is challenging due to the huge number of the degrees of freedom required and the highly indefinite nature of the discrete problem [1]. The proposed Double STDDM can be used as an efficient preconditioner for the problems with constant and heterogenous wavenumbers: for problems with low velocity contrast, the iteration number is independent of the wave number and mesh size, whereas for problems with high velocity contrast, double STDDM can be used as a better preconditioner than the traditional STDDM proposed by the author in [2].

1.1 From STDDM to Double STDDM

We first briefly introduce the STDDM proposed in [2]. Let $\Omega_i=\{x=(x_1,x_2)^T\in\mathbb{R}^2: \zeta_i< x_2<\zeta_{i+1}\}$, $\zeta_i\in\mathbb{R}$, $i=1,\cdots,N$, be the layers whose union covers the support of the source f. Let $\Omega_0=\{x=(x_1,x_2)^T\in\mathbb{R}^2: x_2<\zeta_1\}$ and $\Omega_{N+1}=\{x=(x_1,x_2)^T\in\mathbb{R}^2: x_2>\zeta_{N+1}\}$. Let f_i be the restriction of f in Ω_i and vanish outside Ω_i. It is clear that

$$u(x)=-\int_{\mathbb{R}^2} f(y)G(x,y)dy=-\sum_{i=1}^{N}\int_{\Omega_i} f_i(y)G(x,y)dy,$$

where $G(x,y)$ is the fundamental solution of (1.1) - (1.2):

$$\Delta G(x,y)+k^2(x)G(x,y)=-\delta_y(x) \quad \text{in} \quad \mathbb{R}^2.$$

For the problems with a constant wavenumber, i.e. $k(x)=k$, $G(x,y)=(\mathbf{i}/4)H_0^{(1)}(k|x-y|)$. Let $\bar{f}_1=f_1$. The key idea of the STDDM proposed in [2] is that if one can find a proper way to transfer the source from Ω_i to Ω_{i+1} in the sense that

$$\int_{\Omega_i}\bar{f}_i(y)G(x,y)dy=\int_{\Omega_{i+1}}\Psi_{i+1}(\bar{f}_i)(y)G(x,y)dy, \quad \forall x\in\Omega_j, \quad j>i+1, \tag{1.3}$$

then for $\bar{f}_{i+1}=f_{i+1}+\Psi_{i+1}(\bar{f}_i)$, we transfer sources layer by layer and have

$$u(x)=-\int_{\Omega_N} f_N(y)G(x,y)dy-\int_{\Omega_{N-1}}\bar{f}_{N-1}(y)G(x,y)dy, \quad \forall x\in\Omega_N. \tag{1.4}$$