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Semi-Implicit Spectral Deferred Correction Method Based on the Invariant Energy Quadratization Approach for Phase Field Problems

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Abstract. This paper presents a high order time discretization method by combining the semi-implicit spectral deferred correction method with energy stable linear schemes to simulate a series of phase field problems. We start with the linear scheme, which is based on the invariant energy quadratization approach and is proved to be linear unconditionally energy stable. The scheme also takes advantage of avoiding nonlinear iteration and the restriction of time step to guarantee the nonlinear system uniquely solvable. Moreover, the scheme leads to linear algebraic system to solve at each iteration, and we employ the multigrid solver to solve it efficiently. Numerical results are given to illustrate that the combination of local discontinuous Galerkin (LDG) spatial discretization and the high order temporal scheme is a practical, accurate and efficient simulation tool when solving phase field problems. Namely, we can obtain high order accuracy in both time and space by solving some simple linear algebraic equations.

AMS subject classifications: 65M60, 35L75, 35G25

Key words: Phase field problems, local discontinuous Galerkin method, linear scheme, invariant energy quadratization approach, semi-implicit spectral deferred correction method.

1 Introduction

The phase field model has been used successfully for modeling a variety of interfacial phenomena like microstructure evolution [3] and the physics of phase transitions [16]. The key idea of phase field model is to replace the sharp interface by a thin transition lays, it takes two distinct values (for instance, $+1$ and -1) in each of the phases, with

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a smooth change between both values in the zone around the interfaces, which is then diffused with a finite width.

In the phase field model, the dynamics of the underlying physical system is generally described by a gradient flow resulting from the Euler-Lagrangian variation of a pre-defined energy form with embedded phase-field functions. However, the gradient flow, in the form of high order stiff partial differential equations (PDEs), poses a great deal of difficulty for numerical simulations [24, 39]. Namely, high resolution simulation is preferred in order to capture the generally sharp interfacial structures and to provide numerical solution with fidelity. In addition, the model itself experiences long time evolution therefore computational efficiency is essential to map out the whole dynamics from initial state to steady state.

Various numerical simulations have been developed for phase field problems recently, especially for temporal discretization, which mostly are the convex splitting schemes [4, 10, 12–14, 18, 19, 21, 23, 27, 28], the stabilized linear schemes [11, 17, 20] and others [1, 29]. In the convex splitting scheme, one first splits the energy into convex and nonconvex parts. Then one discretizes the terms of the variational derivative implicitly for the convex part, and explicitly for the nonconvex part of the energy respectively. The resulting convex splitting schemes are unconditionally energy stable. However, they are nonlinear in most cases, which results in nonlinear systems, hence iterative methods are necessary. Moreover, as for the unique solvability of the nonlinear scheme, the proof is not easy. In some cases, nonlinear schemes also require very small time step to guarantee the unique solvability. While for the stabilized linear scheme, the term from the nonlinear potential is simply treated explicitly and some linear stabilizing terms are added to improve the stability. The resulting linear schemes are simple and easy to implement. However, the energy stability depends on the boundedness of the numerical solution, which is not satisfied for most phase field models. In such case, the stabilized linear scheme fails to preserve the energy stability.

In order to overcome the difficulties mentioned above, one hopes to construct a linear scheme which preserves the energy stability. Yang *et al.* developed a novel scheme based on the invariant energy quadratization (IEQ) approach and have been successfully used to solve the molecular beam epitaxial (MBE) growth model [2, 34], the phase field crystal model [32] and various gradient flows [31, 33, 35–38]. The numerical scheme using the IEQ approach takes the following advantages: 1) It is unconditionally energy stable. 2) It is easy to construct the second order scheme, thus more accurate. 3) It is linear and thus easy to implement and efficient. Inspired by the idea, the IEQ approach can be applied to solve other phase field problems automatically, for example, the Allen-Cahn (AC) equation, the Cahn-Hilliard (CH) equation and the Cahn-Hilliard-Hele-Shaw (CHHS) system. The linear scheme is unconditionally energy stable and easy to implement. However, it is only first order or second order accurate in time, and not straightforward to extend to higher order ones. In this paper, we will apply the semi-implicit spectral deferred correction (SDC) method [15] to improve the temporal accuracy, which borrows the linear unconditionally stable method as the basis scheme. We only pay attention to some clas-