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On the Instabilities and Transitions of the Western Boundary Current

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Abstract. We study the stability and dynamic transitions of the western boundary currents in a rectangular closed basin. By reducing the infinite dynamical system to a finite dimensional one via center manifold reduction, we derive a non-dimensional transition number that determines the types of dynamical transition. We show by careful numerical evaluation of the transition number that both continuous transitions (supercritical Hopf bifurcation) and catastrophic transitions (subcritical Hopf bifurcation) can happen at the critical Reynolds number, depending on the aspect ratio and stratification. The regions separating the continuous and catastrophic transitions are delineated on the parameter plane.

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Key words: Western boundary current, dynamic transition, instability, Hopf bifurcation, spectral method.

1 Introduction

Wind-driven circulation/currents and their variability is a central theme in the study of climate dynamics and oceanography. See [10] for a recent comprehensive review of the survey of wind-driven circulation from the perspective of dynamical systems theory. One of the strongest near-surface, mid-latitude currents is the Gulf stream [28]. This current intensifies along the western shores of the North Atlantic (east coast of North America), exhibiting boundary layer characteristics with an intense crowding of the streamlines, and is commonly referred to as the western boundary current. The western boundary currents, which are of intense shear flows, are subject to internal instabilities. In this

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article, we aim to theoretically and numerically study the variability and dynamical transitions of the western boundary currents in a rectangular closed basin.

The study of the western boundary currents has a long history. Earlier works on this topic are mainly done by physicists relying on either analytical solutions of simplified models or purely numerical simulation. Stommel first studies the western boundary currents in the linear regime and discovers that the variation of the Coriolis effect with latitude is the physical mechanism behind the intense currents. Munk in [20] proposes another linear model and finds an analytical solution known as the Munk boundary layer profile that represents a concentrated current like the Gulf stream. The boundary layer analysis of a nonlinear model and the numerical integration can be found in [2, 29, 30] by Veronis. These early results, cf. in particular [4], already suggest that, for the Rossby number in the geophysical regime, there is a critical Reynolds number below which a steady-state solution is approached asymptotically, while shear flow instabilities develop after the transition.

Ierley and Young in [11] study the linear instabilities of a family of basic states including the Munk profile in the western boundary currents by numerically solving a modified Orr-Sommerfeld equation. It is found that the critical Reynolds number at which the basic states become unstable is relatively low (between 20 to 100) and that the unstable modes are trapped in the boundary layer with a slowly varying oscillatory tail. A systematic numerical investigation of the linear instabilities of the western boundary current is carried out by Berloff and Meacham in [3], and by Berloff and McWilliams in [1] for both no-slip and free-slip boundary conditions. Their numerical method is based on the theory of dynamical systems that a change in the stability properties of a steady-state solution will lead to a change in the nature of the solution to which the model asymptotes at a considerable time. By examining carefully the asymptotics of the numerical solution at varying Reynolds number, they only find supercritical Hopf bifurcations in which the amplitudes of the bifurcated limit cycles tend to zero as the critical Reynolds number is approached from the unstable side, though in reality, as we show in this article, subcritical Hopf bifurcations are also possible for certain parameter values. Similar numerical results are also reported in [12, 13]. Mathematically, the existence of bifurcating periodic solutions in a quasi-geostrophic model of wind-driven circulation is investigated by Chen and Price in [6], see also [5]. We refer to the review articles [8, 10] and references therein for the detailed bifurcation analysis of general circulation models.

Previous works on the instabilities of the western boundary currents and the resulting formation of periodic vortices are limited by the scope of the numerical investigation in the presence of multiple control parameters. In this article, we tackle this problem from the perspective of the dynamic transition theory developed by Ma and Wang [18] which is entirely different from previous researches. The main philosophy of this theory is to search for the full set of transition states, giving a complete characterization of stability and transition. The set of transition states can be represented by a local attractor. Following this philosophy, the dynamic transition theory aims to identify the transition states and to classify them both dynamically and physically. One important ingredient