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Central WENO Subcell Finite Volume Limiters for ADER Discontinuous Galerkin Schemes on Fixed and Moving Unstructured Meshes

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Abstract. We present a novel *a posteriori* subcell finite volume limiter for high order discontinuous Galerkin (DG) finite element schemes for the solution of nonlinear hyperbolic PDE systems in multiple space dimensions on fixed and moving unstructured simplex meshes. The numerical method belongs to the family of high order fully discrete one-step ADER-DG schemes [12, 45] and makes use of an element-local space-time Galerkin finite element predictor. Our limiter is based on the MOOD paradigm, in which the discrete solution of the high order DG scheme is checked *a posteriori* against a set of physical and numerical admissibility criteria, in order to detect spurious oscillations or unphysical solutions and in order to identify the so-called troubled cells. Within the detected troubled cells the discrete solution is then *discarded* and *recomputed* locally with a more robust finite volume method on a fine subgrid.
In this work, we propose for the first time to use a high order ADER-CWENO finite volume scheme as subcell finite volume limiter on unstructured simplex meshes, instead of a classical second order TVD scheme. Our new numerical scheme has been developed both for fixed Eulerian meshes as well as for moving Lagrangian grids. It has been carefully validated against a set of typical benchmark problems for the compressible Euler equations of gas dynamics and for the equations of ideal magnetohydrodynamics (MHD).

AMS subject classifications: 65Mxx, 65Zxx

Key words: Discontinuous Galerkin (DG) methods, high order ADER schemes, subcell finite volume limiter, central WENO reconstruction (CWENO), Arbitrary-Lagrangian-Eulerian schemes, moving unstructured meshes, hyperbolic conservation laws, Euler and MHD equations.

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1 Introduction

Nonlinear systems of hyperbolic conservation laws are widely used to model physical systems ranging from gas-dynamics, plasma dynamics, free-surface flows as well as traffic and pedestrian flows. The main difficulty in the numerical solution of these equations arises from the natural appearance of discontinuities after finite times, even when starting from perfectly smooth initial data. This may induce significant spurious oscillations in high order numerical schemes. The governing equations can generally be solved using either an Eulerian or a Lagrangian approach. In the first case the flow is observed and computed in a fixed reference system, while in the latter case the reference system is moving together with the local fluid velocity. In this paper we deal with both Eulerian and Lagrangian schemes, namely using a so-called Arbitrary-Lagrangian-Eulerian (ALE) algorithm [8, 10, 11, 96], in which the mesh velocity can be chosen *independently* from the local fluid velocity, therefore the grid nodes can be arbitrarily moved.

A quite popular numerical method for the solution of hyperbolic systems of balance laws is the Discontinuous Galerkin (DG) approach [2, 12, 24, 26–29, 45, 108, 109], in which the solution is represented by means of a polynomial of some fixed degree in each computational cell. The time evolution of this representation can be performed either using Runge-Kutta time stepping or adopting a one-step ADER approach. ADER [5, 20, 45, 47, 85, 104, 105] is the abbreviation for "Arbitrary high order schemes using DERivatives" and the basic idea is to use the high order reconstructed states, which are available from the reconstruction operator, to evaluate the numerical fluxes at element interfaces. Finite element algorithms for Lagrangian hydrodynamics and the equations of nonlinear elasto-plasticity have been proposed in [40–42, 87, 97], while Lagrangian DG methods have been presented for the first time in [57–59, 75].

DG methods yield very high order accurate and cost-effective schemes, which however suffer from oscillations induced by the Gibbs phenomenon around shock waves and other non-regular features of the solution. It is thus mandatory to introduce some sort of limiting procedure to control these oscillations to make the scheme robust. Classical limiters involve some sort of post-processing of the computed solution at the end of each time-step in order to smooth out the oscillations that have arisen; this must be done in a non-linear way in order to apply this procedure only locally in space, based on the solution itself. For a non-exhaustive overview of limiters for high order DG schemes, see e.g. [29, 38, 39, 77, 90, 91].

At the opposite end of the spectrum of numerical schemes there are the classical first order Godunov-type finite volume methods [100] that are based on the evolution of cell averages of the conserved variables. They require on each face of the computational grid an exact or approximate Riemann solver, see [106] for details, which computes the numerical flux across the element interface in terms of the cell averages of the adjacent cells. Despite their robustness, these schemes are quite diffusive and one would need prohibitively fine grids to solve a realistic problem. Since linear schemes in the sense of Godunov cannot exceed first order accuracy [100], non-oscillatory second and higher