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# Theory of Spontaneous Symmetry Breaking and an Application to Superconductivity: Nambu-Goldstone and Higgs Excitation Modes

Takashi Yanagisawa*

*Electronics and Photonics Research Institute, National Institute of Advanced Industrial Science and Technology, 1-1-1 Umezono, Tsukuba, Ibaraki 305-8568, Japan.*

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**Abstract.** We present a general framework of the theory of spontaneous symmetry breaking in non-relativistic systems. We discuss a spontaneous symmetry breaking in a system with general global symmetry given by a Lie group $G$. The Nambu-Goldstone boson and Higgs boson are represented explicitly by local fields by means of the basis of the Lie algebra of $G$. An application to superconductivity is discussed. We evaluate the Green's functions of the Nambu-Goldstone and Higgs bosons in superconductors. We show that the Nambu-Goldstone and Higgs modes exhibit interesting behaviors in multi-component superconductors.

**AMS subject classifications**: 22E, 22C, 70H, 70S

**Key words**: Spontaneous symmetry breaking, Lie algebra, Nambu-Goldstone boson, Higgs boson, Ward-Takahashi identity, superconductivity, multi-band superconductor.

## 1 Introduction

When the Lagrangian is invariant under a symmetry transformation, there is a conserved current and a conserved quantity. The spontaneous symmetry breaking indicates that the state is not invariant under s symmetry transformation although the Lagrangian is invariant under this transformation. This occurs when an asymmetric state is realized in a symmetric system. It is well known that when a continuous symmetry is spontaneously broken, a massless boson appears. This boson is called the Nambu-Goldstone boson (NG boson) [1–3]. General proofs of the existence of the NG boson were given in [3, 4]. The spontaneous symmetry breaking has been studied intensively in the condensed-matter physics [5–10] and in field theory [11–21].

*Corresponding author. *Email address:* `t-yanagisawa@aist.go.jp` (T. Yanagisawa)

 

We investigate a general formulation of the Nambi-Goldstone boson in a system with spontaneous symmetry breaking. The model has a symmetry of continuous Lie group $G$. The Nambu-Goldstone boson is represented by means of local fields with bases of Lie algebra of $G$. The existence of the massless boson is shown on the basis of this representation. The Ward-Takahashi identity is modified to include a vertex correction due to the Nambu-Goldstone boson in a spontaneous symmetry broken system. This means that the breaking of the Ward-Takahashi identity owing to spontaneous symmetry breaking is compensated by the inclusion of the Nambu-Goldstone boson. We also discuss applications to physical systems. In particular, we examine the Nambu-Goldstone and Higgs modes in superconductors.

This paper is organized as follows. In Section II, we investigate spontaneous symmetry breaking for a general compact group $G$. We discuss the Ward-Takahashi identity in this section. In Section III, we discuss excitation modes in superconductors. The Section IV is devoted to an investigation of the low energy Nambu-Goldstone and Higgs modes in multi-band superconductors. We give a summary in last section.

## 2 Spontaneous symmetry breaking

### 2.1 Symmetry Group

Let $G$ be a compact Lie group (symmetry group) and $g$ be the Lie algebra of $G$. The elements of the basis set of the Lie algebra $g$ as $T_a$ ($a$=$1,\cdots,N_G$) where $N_G$ is the dimension of $G$. We adopt that the fermion field $\psi$ is transformed under the action of the Lie group $G$ as

$$\psi \to e^{-i\theta T_a}\psi = \psi - i\theta T_a\psi + \mathcal{O}(\theta^2), \tag{2.1}$$

where $\theta$ is an infinitesimal parameter. We put $\delta\psi = -i\theta T_a\psi$. $\{T_a\}$ are normalized as

$$\mathrm{Tr} T_a T_b = c\delta_{ab}, \tag{2.2}$$

where $c$ is a real constant. The structure constants are introduced through commutation relations,

$$[T_a, T_b] = \sum_c i f_{abc} T_c. \tag{2.3}$$

When the Lagrangian $\mathcal{L}$ is invariant under the transformation $\psi \to \psi + \delta\psi$, there is a conserved current. We denote the current for the transformation generated by $T_a$ as $j_a^\mu$:

$$j_a^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\delta\psi. \tag{2.4}$$

This satisfies $\partial_\mu j_a^\mu = 0$. The conserved quantities are defined by

$$Q_a = \int d\mathbf{r} J_a^0(\mathbf{r}), \tag{2.5}$$