

REVIEW ARTICLE

Mathematical Models and Numerical Methods for Spinor Bose-Einstein Condensates

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Received 1 September 2017; Accepted (in revised version) 8 December 2017

Abstract. In this paper, we systematically review mathematical models, theories and numerical methods for ground states and dynamics of spinor Bose-Einstein condensates (BECs) based on the coupled Gross-Pitaevskii equations (GPEs). We start with a pseudo spin-1/2 BEC system with/without an internal atomic Josephson junction and spin-orbit coupling including (i) existence and uniqueness as well as non-existence of ground states under different parameter regimes, (ii) ground state structures under different limiting parameter regimes, (iii) dynamical properties, and (iv) efficient and accurate numerical methods for computing ground states and dynamics. Then we extend these results to spin-1 BEC and spin-2 BEC. Finally, extensions to dipolar spinor systems and/or general spin- F ($F \geq 3$) BEC are discussed.

AMS subject classifications: 35Q55, 35P30, 65M06, 65M70, 65Z05, 81Q05

Key words: Bose-Einstein condensate, Gross-Pitaevskii equation, spin-orbit, spin-1, spin-2, ground state, dynamics, numerical methods.

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1 Introduction

The remarkable experimental achievement of Bose-Einstein condensation (BEC) of dilute alkali gases in 1995 [4,44,63] reached a milestone in atomic, molecular and optical (AMO) physics and quantum optics, and it provided a unique opportunity to observe the mysterious quantum world directly in laboratory. The BEC phenomenon was predicted by Einstein in 1924 [66,67] when he generalized the studies of Bose [43] concerning photons to atoms which assume the same statistical rule. Based on the derived Bose-Einstein statistics, Einstein figured that, there exists a critical temperature, below which a finite fraction of all the particles “condense” into the same quantum state.

Einstein’s prediction was for a system of noninteracting bosons and did not receive much attention until the observation of superfluidity in liquid ^4He below the λ temperature (2.17K) in 1938, when London [98] suggested that despite the strong interatomic interactions, part of the system is in the BEC state resulting in its superfluidity. Over the years, the major difficulty to realize BEC state in laboratory is that almost all the substances become solid or liquid (strong interatomic interactions) at low temperature where the BEC phase transition occurs. With the development of magnetic trapping and laser cooling techniques, BEC was finally achieved in the system of weakly interacting dilute alkali gases [4,44,63] in 1995. The key is to bring down the temperature of the gas before its relaxation to solid state. In most BEC experiments, the system reaches quantum degeneracy between 50 nK and 2 μK , at densities between 10^{11} and 10^{15} cm^{-3} . The largest condensates are of 100 million atoms for sodium, and a billion for hydrogen; the smallest are just a few hundred atoms. Depending on the magnetic trap, the shape of the condensate is either approximately round, with a diameter of 10–15 μm , or cigar-shaped with about 15 μm in diameter and 300 μm in length. The full cooling cycle that produces a condensate may take from a few seconds to as long as several minutes [59,86]. For better understanding of the long history towards the BEC and its physics study, we refer to the Nobel lectures [59,86] and several review papers [38,42,60,69,88,90] as well as the two books [109,111] in physics.

The pioneering experiments [4,44,63] were conducted for single species of atoms, which can be theoretically described by a scalar order parameter (or wavefunction) satisfying the Gross-Pitaevskii equation (GPE) (or the nonlinear Schrödinger equation (NLSE) with cubic nonlinearity) [60,69,90,109,111]. For the mathematical models and numerical methods of single-component BEC based on the GPE, we refer to [3,6,11,15,64,68,77,82,92] and references therein. A natural generalization is to explore the multi-component BEC system, where inter-species interactions lead to more interesting phases and involve vector order parameters. In 1996, one year after the major breakthrough, an overlapping two component BEC was produced with $|F=2, m=2\rangle$ and $|F=1, m=-1\rangle$ spin states of ^{87}Rb [106], by employing a double magneto-optic trap. During the process, two condensates were cooled together and the interaction between different components was observed. Later, it was proposed that the binary BEC system can generate coherent matter wave (also called atom laser) analogous to the coherent light emitted from a laser. In view