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Construction of Symplectic Runge-Kutta Methods for Stochastic Hamiltonian Systems

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Abstract. We study the construction of symplectic Runge-Kutta methods for stochastic Hamiltonian systems (SHS). Three types of systems, SHS with multiplicative noise, special separable Hamiltonians and multiple additive noise, respectively, are considered in this paper. Stochastic Runge-Kutta (SRK) methods for these systems are investigated, and the corresponding conditions for SRK methods to preserve the symplectic property are given. Based on the weak/strong order and symplectic conditions, some effective schemes are derived. In particular, using the algebraic computation, we obtained two classes of high weak order symplectic Runge-Kutta methods for SHS with a single multiplicative noise, and two classes of high strong order symplectic Runge-Kutta methods for SHS with multiple multiplicative and additive noise, respectively. The numerical case studies confirm that the symplectic methods are efficient computational tools for long-term simulations.

AMS subject classifications: 65C30, 60H35, 65P10

Key words: Stochastic differential equation, Stochastic Hamiltonian system, symplectic integration, Runge-Kutta method, order condition.

1 Introduction

Consider the following Cauchy problem for stochastic differential equations (SDEs):

$$dX_t = a(t,X_t)dt + \sum_{k=1}^{m} b_k(t,X_t) * dw_t^k, \qquad X_{t_0} = x_0, \tag{1.1}$$

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where $X, a(t,x^1,\cdots,x^r)$, $b_k(t,x^1,\cdots,x^r)$ are r-dimensional column-vectors with the components X^i, a^i, b^i_j, $i=1,\cdots,r$, $a,b_k \in C^{2\eta}(\mathbb{R}^r,\mathbb{R}^r)$, $\eta=1,2,\cdots$, and where w^k_t, $k=1,\cdots,m$, are independent standard Wiener processes. We write $*dw^k_t = dw^k_t$ in the case of an Itô stochastic integral and $*dw^k_t = \circ dw^k_t$ for a Stratonovich stochastic integral.

Let us write a system of SDEs of even dimension $r=2d$ in the form of stochastic Hamiltonian systems (SHS) in the sense of Stratonovich:

$$\begin{aligned} dP^i &= -\frac{\partial H_0(t,P,Q)}{\partial Q^i}dt - \sum_{k=1}^{m} \frac{\partial H_k(t,P,Q)}{\partial Q^i} \circ dw^k_t, \qquad P(t_0)=p, \\ dQ^i &= \frac{\partial H_0(t,P,Q)}{\partial P^i}dt + \sum_{k=1}^{m} \frac{\partial H_k(t,P,Q)}{\partial P^i} \circ dw^k_t, \qquad Q(t_0)=q \end{aligned} \tag{1.2}$$

for $d,m \geq 1$ with an m-dimensional Wiener process $(w_t)_{t\geq 0}$ and $t \in \mathbb{R}$, where P,Q,p,q are d-dimensional vectors with components P^i, Q^i, p^i, q^i, $i=1,2,\cdots,d$. The SHS (1.2) includes both Hamiltonian systems with additive or multiplicative noise.

For SHS (1.2), [28] established the theory about the stochastic symplectic methods which preserve the symplectic structure of the SDEs. Tretyakov and Tret'jakov [40] considered numerical methods for Hamiltonian systems with external noise. Seesselberg et al. [38] investigated the numerical simulation of singly noisy Hamiltonian systems and their application to particle storage rings. Misawa [29] proposed an energy conservative stochastic difference scheme for a one-dimensional stochastic Hamilton dynamical system. Milstein, Repin and Tretyakov [25,26] investigated symplectic integration of SHS (1.2) with additive and multiplicative noise, respectively. Hong, Scherer and Wang [14,15] investigated numerical methods for linear stochastic oscillator with additive noise. Milstein and Tretyakov [27] presented quasi-symplectic integration for Langevin-type equations. Wang et al [41, 42] discussed variational integrators and generating functions of SHS (1.2). Deng, Anton and Wong [12] proposed some high order symplectic schemes based on generating functions. Abdulle, Cohen, Vilmart and Zygalakis [1] proposed a new methodology for constructing numerical integrators with high weak order for the time integration of stochastic differential equations based on modified equations. Hong, Zhai and Zhang [17] proposed discrete gradient approach to stochastic differential equations with a conserved quantity. Cohen and Duardin [8] proposed a new class of energy-preserving numerical schemes for stochastic Hamiltonian systems with noncanonical structure matrix in the Stratonovich sense. Hong, Xu and Wang [16] investigated quadratic invariant-preserving SRK methods for SDEs possessing an invariant in the sense of Stratonovich. Recently, Cristina, Deng and Wong [9, 10] discussed symplectic schemes for SHS and stochastic systems preserving Hamiltonian functions, respectively. Using generating functions, Wang [42] presented the generalization of a symplectic Runge-Kutta method for SHS with a single noise in the sense of Stratonovich. Ma, Ding and Ding [23] presented the symplectic conditions of SRK methods for SHS with a single noise in the sense of Stratonovich. And the above two works are concerned about the strong convergence case. Here we will discuss the more general cases that in-