Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0033

Vol. **21**, No. 1, pp. 16-39
January 2017

Artificial Boundary Conditions for Nonlocal Heat Equations on Unbounded Domain

Wei Zhang[1,*], Jiang Yang[2], Jiwei Zhang[1] and Qiang Du[2]

[1] *Beijing Computational Science Research Centre, Beijing, P.R. China.*
[2] *Department of Mathematics, Columbia University, New York, NY 10027, USA.*

Received 1 March 2016; Accepted (in revised version) 26 May 2016

Abstract. This paper is concerned with numerical approximations of a nonlocal heat equation define on an infinite domain. Two classes of artificial boundary conditions (ABCs) are designed, namely, nonlocal analog Dirichlet-to-Neumann-type ABCs (global in time) and high-order Padé approximate ABCs (local in time). These ABCs reformulate the original problem into an initial-boundary-value (IBV) problem on a bounded domain. For the global ABCs, we adopt a fast evolution to enhance computational efficiency and reduce memory storage. High order fully discrete schemes, both second-order in time and space, are given to discretize two reduced problems. Extensive numerical experiments are carried out to show the accuracy and efficiency of the proposed methods.

AMS subject classifications: 45A05, 65D99, 65R20

Key words: Artificial boundary conditions, nonlocal models, Padé approximation, nonlocal heat equations, artificial boundary method.

1 Introduction

In this paper, we consider the numerical computation of 1D nonlocal heat equations on an unbounded spatial domain, given by

$$u_t(x,t)=\mathcal{L}_\delta u(x,t)+f(x,t), \quad (x,t)\in\mathbb{R}\times(0,T], \tag{1.1}$$
$$u(x,0)=g(x), \quad x\in\mathbb{R}, \tag{1.2}$$
$$u(x,t)\to 0, \quad \text{as } |x|\to\infty,\ \forall t>0, \tag{1.3}$$

*Corresponding author. *Email addresses:* `wzhang@csrc.ac.cn` (W. Zhang), `jyanghkbu@gmail.com` (J. Yang), `jwzhang@csrc.ac.cn` (J. Zhang), `qd2125@columbia.edu` (Q. Du)

where the initial value $g = g(x)$ and the source $f = f(x,t)$ are the given compactly supported functions. The nonlocal operator $\mathcal{L}_\delta$ is defined as

$$\mathcal{L}_\delta u(x) = \int_{\mathbf{B}_\delta(x)} (u(x') - u(x))\gamma_\delta(x,x')dx', \qquad \forall x \in \mathbb{R}^d,$$

where $\mathbf{B}_\delta(x) = \{x' | x' \in \mathbb{R}^d : |x' - x| < \delta\}$ is a neighborhood of x with radius δ. Usually, $\gamma_\delta = \gamma_\delta(x,x') : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a nonnegative, radial-type kernel, namely, $\gamma_\delta(x,x') = \gamma_\delta(|x - x'|)$. Furthermore, it is compactly supported by $x' \notin \mathbf{B}_\delta(x)$ and satisfies the following second moment condition

$$0 < \int_0^\delta \tau^2 \gamma_\delta(\tau) d\tau = C_\delta < \infty. \tag{1.4}$$

Eq. (1.3) may be viewed as a nonlocal-in-space analog of the the classical heat equations [13]. Although the latter have been widely applied in many fields based on Fick's first law for local diffusive fluxes [4, 36, 37], nonlocal heat equations offer better models for anomalous diffusion behavior. Nonlocal integral operators in the form of $\mathcal{L}_\delta$ have also been used in nonlocal peridynamics models of mechanics [12, 18, 23, 39–41], thermal diffusion and electromigration [3, 7–9, 15, 17, 20]. A mathematical framework of nonlocal vector calculus and nonlocal balance laws has been developed in [14, 22]. It has been applied to study related volume-constraint problem [13, 14, 34]. Aside from mathematical analysis of PD/nonlocal models, there are also various numerical methods such as finite difference, finite element, quadrature and particle-based methods [10,16,33,49]. Recently, Tian and Du [43, 44] present the deep insight for the numerical approximations to nonlocal models. Most importantly, their works address the issue of convergence in both the nonlocal setting and the local limit, and find the asymptotically compatible schemes for nonlocal models. All the above PD/nonlocal simulations focus on numerically solving problems with fixed boundary conditions on bounded domain. In fact, there are many applications in which the simulation of an infinite medium may be useful, such as wave or crack propagation, superdiffusion in a whole space.

The nonlocal heat equations/peridynamic thermal diffusion models under consideration can be formulated by the nonlocal heat transfer between material points [7, 8]. In one dimensional space, the nonlocal operator becomes

$$\mathcal{L}_\delta u(x) = \int_{-\delta}^{\delta} [u(x+\tau) - u(x)]\gamma_\delta(\tau)d\tau. \tag{1.5}$$

With a suitably defined kernel, as $\delta \to 0$, for a smooth function $u = u(x)$, we may have $\mathcal{L}_\delta u \to a^2 d^2 u/dx^2$ for some constant $a > 0$ [13, 34]. Thus, the nonlocal model (1.1)-(1.3) in the local limit tends to the classical heat equation

$$u_t(x,t) = a^2 u_{xx}(x,t) + f(x,t), \quad (x,t) \in \mathbb{R} \times (0,T], \tag{1.6}$$

$$u(x,0) = g(x), \quad x \in \mathbb{R}, \tag{1.7}$$

$$u \to 0, \quad \text{as } |x| \to \infty. \tag{1.8}$$