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A Mathematical Analysis of Scale Similarity

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Abstract. Scale similarity is found in many natural phenomena in the universe, from fluid dynamics to astrophysics. In large eddy simulations of turbulent flows, some sub-grid scale (SGS) models are based on scale similarity. The earliest scale similarity SGS model was developed by Bardina et al., which produced SGS stresses with good correlation to the true stresses. In the present study, we perform a mathematical analysis of scale similarity. The analysis has revealed that the ratio of the resolved stress to the SGS stress is γ^2, where γ is the ratio of the second filter width to the first filter width, under the assumption of small filter width. The implications of this analysis are discussed in the context of large eddy simulation.

AMS subject classifications: 35L65, 76F02

Key words: Scale similarity, large eddy simulation, sub-grid scale models.

1 Introduction

After decades of development, large eddy simulations (LES) [5, 14, 19] are being used in the design process to predict vortex dominated, highly separated turbulent flows found in many important applications in aerospace, mechanical and chemical engineering. One important challenge in LES is the determination of the sub-grid scale (SGS) stress resulted from the filtering of the nonlinear governing Navier-Stokes equations. Various models have been developed to compute the SGS stress from the resolved field variables. Popular models comprise the Smagorinsky model (SM) and its many variants including the static [19] and dynamic SM [8], the scale similarity model (SSM) [1], the mixed model of SSM and SM [1], and monotone integrated LES [2] or implicit LES [9], in which no explicit SGS model is used. There is an extensive literature on each model with many applications.

Since the pioneering work by Bardina et al. on the SSM [1] for incompressible flow, there has been an extensive effort in evaluating its performance by comparing with other SGS models [7, 12, 16]. Furthermore the SSM has been extended to other flow problems

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including compressible flow [6] and combustion [11]. Direct numerical simulations and experimental measurements [15, 16, 18] have demonstrated scale similarity in turbulent flows. Many a priori tests using experimental or DNS data have shown a high correlation between the true stress and the modeled SGS stress based on the SSM [15]. In Bardina's original SSM, the second filter or test filter has the same width as the first one. It was proven by Speziale [20] that the Bardina constant must be 1 to satisfy Galilean invariance. The present analysis to be shown later confirms this result. Other researchers suggested using a different filter width for the second filter, e.g., in [15], and many approaches were suggested to determine the Bardina constant [4, 18].

In a recent effort to understand the performance of these SGS models in the context of discontinuous high order methods such as the discontinuous Galerkin method [3] or the flux reconstruction (FR)/correction procedure via reconstruction method (CPR) [10, 21], we performed a priori and a posteriori studies using the 1D Burgers' equation [13]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},\qquad x\in[-1,1], \tag{1.1}$$

where u is the state variable, ν is a constant viscosity, chosen to be 8×10^{-5} in the present study to mimic high Reynolds number flow problems. The LES governing equation is then obtained by filtering (1.1) with a low pass filter $G_\Delta(x,\xi)$ satisfying the following conservative property

$$\int_{-\infty}^{+\infty} G_\Delta(x,\xi)d\xi=1, \tag{1.2}$$

where Δ denotes the filter width. Typical filters include the top hat filter defined as

$$G_\Delta(x,\xi)=\begin{cases} 1/\Delta, & |x-\xi|\leqslant\Delta/2,\\ 0, & \text{otherwise},\end{cases} \tag{1.3}$$

and the Gaussian filter

$$G_\Delta(x,\xi)=\sqrt{\frac{6}{\pi\Delta^2}}e^{-\frac{6(x-\xi)^2}{\Delta^2}}. \tag{1.4}$$

The filtering process is defined mathematically in the physical space as a convolution operator. The filtered variable $\hat{\phi}(x,t)$ of a space-time variable $\phi(x,t)$ in 1D is defined as

$$\hat{\phi}(x,t)=\int_{-\infty}^{+\infty} G_\Delta(x,\xi)\phi(\xi,t)d\xi. \tag{1.5}$$

The filtering process is linear, i.e, $\widehat{\phi+\varphi}=\hat{\phi}+\hat{\varphi}$. If the filter width is constant, the differential and the filter operators commute, i.e., $\widehat{\frac{\partial\phi}{\partial x}}=\frac{\partial\hat{\phi}}{\partial x}$. Applying a low-pass spatial filter to Eq. (1.1), we obtain the following filtered equation

$$\frac{\partial\hat{u}}{\partial t}+\hat{u}\frac{\partial\hat{u}}{\partial x}=\nu\frac{\partial^2\hat{u}}{\partial x^2}-\frac{1}{2}\frac{\partial\tau}{\partial x}, \tag{1.6}$$