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A New Coupled Complex Boundary Method for Bioluminescence Tomography

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Abstract. In this paper, we introduce and study a new method for solving inverse source problems, through a working model that arises in bioluminescence tomography (BLT). In the BLT problem, one constructs quantitatively the bioluminescence source distribution inside a small animal from optical signals detected on the animal's body surface. The BLT problem possesses strong ill-posedness and often the Tikhonov regularization is used to obtain stable approximate solutions. In conventional Tikhonov regularization, it is crucial to choose a proper regularization parameter for trade off between the accuracy and stability of approximate solutions. The new method is based on a combination of the boundary condition and the boundary measurement in a parameter-dependent single complex Robin boundary condition, followed by the Tikhonov regularization. By properly adjusting the parameter in the Robin boundary condition, we achieve two important properties for our new method: first, the regularized solutions are uniformly stable with respect to the regularization parameter so that the regularization parameter can be chosen based solely on the consideration of the solution accuracy; second, the convergence order of the regularized solutions reaches one with respect to the noise level. Then, the finite element method is used to compute numerical solutions and a new finite element error estimate is derived for discrete solutions. These results improve related results found in the existing literature. Several numerical examples are provided to illustrate the theoretical results.

AMS subject classifications: 65N21, 65F22, 49J40, 74S05

Key words: Bioluminescence tomography, Tikhonov regularization, convergence rate, finite element methods, error estimate.

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1 Introduction

Bioluminescence tomography (BLT) is a new molecular imaging modality and has shown its potential in monitoring non-invasively physiological and pathological processes *in vivo* at the cellular and molecular level. It is particularly attractive for *in vivo* applications because no external excitation source is needed and thus background noise is low while sensitivity is high [21]. In the BLT problem, one reconstructs quantitatively the bioluminescence source distribution inside a small animal from optical signals detected on the animal's body surface. Let $\Omega \subset \mathbb{R}^d$ ($d \leq 3$: space dimension) be an open bounded set with boundary $\Gamma := \partial\Omega$. Then without going into detail, we state the BLT problem as follows.

Problem 1.1. Find a source function p inside Ω so that the solution u of the forward (real) Robin boundary value problem (BVP)

$$\begin{cases} -\mathrm{div}(D\nabla u) + \mu_a u = p & \text{in } \Omega, \\ u + 2AD\frac{\partial u}{\partial n} = g^- & \text{on } \Gamma \end{cases} \tag{1.1}$$

satisfies the outgoing flux density on the boundary:

$$g = -D\frac{\partial u}{\partial n} \quad \text{on } \Gamma_0. \tag{1.2}$$

Here $D = [3(\mu_a + \mu')]^{-1}$ is the diffusion coefficient with μ_a and μ' being known as the absorption and reduced scattering parameters; $\partial/\partial n$ stands for the outward normal derivative; g^- is an incoming flux on Γ and it vanishes when the imaging is implemented in a dark environment; $\Gamma_0 \subset \Gamma$ is the part of the boundary for measurement; $A = A(x) = (1+R(x))/((1-R(x)))$ with $R(x) \approx -1.4399\gamma(x)^{-2} + 0.7099\gamma(x)^{-1} + 0.6681 + 0.0636\gamma(x)$ and $\gamma(x)$ being the refractive index of the medium at $x \in \Gamma$. In what follows, we restrict ourselves to the case where $g^- \equiv 0$ and $\Gamma_0 = \Gamma$.

Inverse source problems with only one measurement on the boundary do not have a unique solution. In the context of the BLT problem, one cannot distinguish between a strong source over a small region and a weak source over a large region. For instance, let Ω be the unit circle centered at the origin, $\mu_a = 0.04$, $\mu' = 1.5$, and $A = 3.2$ with refractive index $\gamma = 1.3924$. We assign two different source functions: a strong small source function $p_1 = 4$ in a circle centered $(0.5,0)$ with radius 0.2 and a weak big source function $p_2 = 1$ in a circle centered $(0.5,0)$ with radius 0.4. Although the solutions u_1 and u_2 of (1.1), corresponding to p_1 and p_2 respectively, differ greatly in Ω, they have almost the same outgoing flux density g on the boundary, as is shown in Fig. 1. This agrees with the theoretical result about the solution non-uniqueness presented in [12]. One can have better identification with more a priori information about the source function p. One of the a priori information is a permissible region $\Omega_0 \subset \Omega$ of the optical source distribution. In this case, the first equation of (1.1) is replaced by

$$-\mathrm{div}(D\nabla u) + \mu_a u = p\chi_{\Omega_0} \quad \text{in } \Omega,$$