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A Fast Solver for an $\mathcal{H}_1$ Regularized PDE-Constrained Optimization Problem

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Abstract. In this paper we consider PDE-constrained optimization problems which incorporate an $\mathcal{H}_1$ regularization control term. We focus on a time-dependent PDE, and consider both distributed and boundary control. The problems we consider include bound constraints on the state, and we use a Moreau-Yosida penalty function to handle this. We propose Krylov solvers and Schur complement preconditioning strategies for the different problems and illustrate their performance with numerical examples.

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1 Introduction

As methods for numerically solving partial differential equations (PDEs) become more accurate and well-understood, some focus has shifted to the development of numerical methods for optimization problems with PDE constraints: see, e.g., [41,44,69] and the references mentioned therein. The canonical PDE-constrained optimization problem takes a given *desired state*, $\bar{y}$, and finds a *state*, y, and a *control*, u, to minimize the functional

$$\|y-\bar{y}\|_{\mathcal{Y}}^2+\frac{\beta}{2}R(u) \tag{1.1}$$

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subject to the constraints

$$\begin{aligned}&\mathcal{A}y=u,\\&u_a\le u\le u_b,\\&y_a\le y\le y_b,\end{aligned}$$

where $\|\cdot\|_{\mathcal{Y}}$ is some norm and $R(u)$ is a regularization functional. We are free to choose both the norm and the regularization functional here; appropriate choices often depend on the properties of the underlying application. In the description above $\mathcal{A}$ denotes a PDE with appropriate boundary conditions and β denotes a scalar regularization parameter. The focus of this manuscript is regularization based on the H_1 norm of the control, which we motivate below.

The simplest choice of $R(u)$ is $\|u\|^2_{L_2(\Omega)}$, where Ω denotes the domain on which the PDE is posed. This case has been well-studied in the literature, both from a theoretical and algorithmic perspective. However, the requirements of real-world problems has necessitated the application of alternative regularization terms.

One area where there has been much interest is in regularization using L_1 norms, see, e.g., the recent articles [12, 73]. A related norm is the total variation norm $R(u)=\|\nabla u\|_{L_1(\Omega)}$, has also aroused excitement recently – see e.g. [14, 59] and the references therein. These L_1 norms have the benefit that they allow discontinuous controls, which can be important in certain applications.

For certain applications it is desirable to have a smooth control – for this reason the $\mathcal{H}_1$ semi-norm, $R(u)=\|\nabla u\|^2_{L_2(\Omega)}$, has long been studied in the context of parameter-estimation problems [10,46,76], image-deblurring [13,17,48], image reconstruction [49], and flow control [18, 34], for example. Recently van den Doel, Ascher and Haber [19] argued that this norm can be a superior choice to its L_1-based cousin, total variation, for problems with particularly noisy data due to the smooth nature of controls which arise. The test problems in PDE constrained optimization by Haber and Hanson [31], which were designed to get academics solving problems more in-line with the needs of the real-world, suggest a regularization functional of the form $R(u)=\|u\|^2_{L_2(\Omega)}+\alpha\|\nabla u\|^2_{L_2(\Omega)}$ for a given α. Indeed, this form of regularization is commonly used in the ill-posed and inverse problem communities. Another example of a field where the standard L_2 regularization may not be appropriate is flow control – see, e.g., Gunzburger [28, Chapter 4].

At the heart of many techniques for solving the optimization problem, whether it is a linear problem or the linearization of a non-linear problem, lies the solution of a linear system [35,41,44,70]. These systems are very often so-called saddle point matrices [4,23], which have the form

$$\mathcal{A}=\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}, \tag{1.2}$$

where A represents the misfit and regularization terms in (1.1) and B represents the PDE constraint. In the systems we consider in this paper, A is symmetric positive semi-definite. Such saddle point matrices are invertible if B has full rank and $\ker(A)\cap\ker(B)=$