A New Approach of High Order Well-Balanced Finite Volume WENO Schemes and Discontinuous Galerkin Methods for a Class of Hyperbolic Systems with Source Terms[†]

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Abstract. Hyperbolic balance laws have steady state solutions in which the flux gradients are nonzero but are exactly balanced by the source terms. In our earlier work [31–33], we designed high order well-balanced schemes to a class of hyperbolic systems with separable source terms. In this paper, we present a different approach to the same purpose: designing high order well-balanced finite volume weighted essentially non-oscillatory (WENO) schemes and Runge-Kutta discontinuous Galerkin (RKDG) finite element methods. We make the observation that the traditional RKDG methods are capable of maintaining certain steady states exactly, if a small modification on either the initial condition or the flux is provided. The computational cost to obtain such a well balanced RKDG method is basically the same as the traditional RKDG method. The same idea can be applied to the finite volume WENO schemes. We will first describe the algorithms and prove the well balanced property for the shallow water equations, and then show that the result can be generalized to a class of other balance laws. We perform extensive one and two dimensional simulations to verify the properties of these schemes such as the exact preservation of the balance laws for certain steady state solutions, the non-oscillatory property for general solutions with discontinuities, and the genuine high order accuracy in smooth regions.

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1 Introduction

In this paper, we are concerned with the construction of high order well balanced weighted essentially non-oscillatory (WENO) finite volume schemes and Runge-Kutta discontinuous Galerkin (RKDG) finite element methods for solving hyperbolic balance laws, which have attracted significant attention in the past few years. Hyperbolic balance laws are hyperbolic systems of conservation laws with source terms:

$$u_t + f_1(u, x, y)_x + f_2(u, x, y)_y = g(u, x, y) \tag{1.1}$$

or in the one dimensional case

$$u_t + f(u, x)_x = g(u, x) \tag{1.2}$$

where u is the solution vector, $f_1(u, x, y)$ and $f_2(u, x, y)$ (or $f(u, x)$) are the fluxes and $g(u, x, y)$ (or $g(u, x)$) is the source term.

An essential part for these balance laws is that they often admit steady state solutions in which the flux gradients are exactly balanced by the source term. A straightforward treatment of the source terms in a numerical scheme will fail to preserve this balance. Many physical phenomena come from small perturbations of these steady state solutions, which are very difficult to capture numerically, unless the numerical schemes can preserve the unperturbed steady state at the discrete level. Schemes which can preserve the unperturbed steady state at the discrete level are the so called well balanced schemes. Our purpose is to design well balanced schemes without sacrificing the high order accuracy and non-oscillatory properties of the scheme when applied to general, non-steady state solutions.

Balance laws have many applications in the physical world. A typical and extensively considered example is the shallow water equations with a non flat bottom topology. Many geophysical flows are modeled by the variants of the shallow water equations. In one space dimension, they take the form

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \dfrac{1}{2}gh^2 \right)_x = -ghb_x, \end{cases} \tag{1.3}$$

where h denotes the water height, u is the velocity of the fluid, b represents the bottom topography and g is the gravitational constant. The steady state solutions are given by

$$hu = constant \qquad and \qquad \frac{1}{2}u^2 + g(h+b) = constant. \tag{1.4}$$

People are particularly interested in the still water stationary solution, denoted by

$$u = 0 \qquad and \qquad h + b = constant. \tag{1.5}$$

Bermudez and Vazquez [4] first introduced the concept of the "exact C-property", which means that the scheme is "exact" when applied to the still water stationary solution (1.5). Also, they introduced the first order Q-scheme and the idea of source term upwinding to obtain such