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Stability of Projection Methods for Incompressible Flows Using High Order Pressure-Velocity Pairs of Same Degree: Continuous and Discontinuous Galerkin Formulations

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Abstract. This paper presents limits for stability of projection type schemes when using high order pressure-velocity pairs of same degree. Two high order *h/p* variational methods encompassing continuous and discontinuous Galerkin formulations are used to explain previously observed lower limits on the time step for projection type schemes to be stable [18], when *h*- or *p*-refinement strategies are considered. In addition, the analysis included in this work shows that these stability limits do not depend only on the time step but on the product of the latter and the kinematic viscosity, which is of particular importance in the study of high Reynolds number flows. We show that high order methods prove advantageous in stabilising the simulations when small time steps and low kinematic viscosities are used.

Drawing upon this analysis, we demonstrate how the effects of this instability can be reduced in the discontinuous scheme by introducing a stabilisation term into the global system. Finally, we show that these lower limits are compatible with Courant-Friedrichs-Lewy (CFL) type restrictions, given that a sufficiently high polynomial order or a mall enough mesh spacing is selected.

AMS subject classifications: 76D05, 35Q30, 65N30

Key words: Incompressible Navier-Stokes equations, projection methods, velocity-correction, continuous Galerkin, discontinuous Galerkin, inf-sup LBB condition.

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1 Introduction

Since the introduction of projection type algorithms for the efficient solution of the incompressible Navier-Stokes equations in primitive variables by Chorin [8] and Temam [35], various modifications have emerged. These variations, whereby intermediate variables enable some degree of decoupling of pressure and velocity, may be viewed as one of three categories: *pressure-correction*, *velocity projection* and *consistent splitting* methods [18]. These methods share the appealing property of requiring the solution of decoupled elliptic equations for the velocity and pressure fields, which renders these type of schemes very efficient and thus extremely useful in numerical simulations.

For many years, it had been thought that this category of schemes do not require an inf-sup condition to be fulfilled since velocity and pressure are essentially "decoupled". The inf-sup (or LBB from Ladyzhenskaya [26], Babushka [3] and Brezzi [5]) condition typically states that an equal order expansion for both pressure and velocity leads to an unstable system [6,13,23]. To overcome this difficulty one can augment the velocity space with respect to the pressure space or add stabilisation whilst maintaining the same space dimensions for pressure and velocity.

Projection-type schemes have historically been implemented using equal order spaces for pressure and velocity since they appeared to be stable in this setting. However, in recent years it has been shown that these schemes instead correspond to stabilised-like schemes [4, 10, 18]. Auteri et al. [2] and Guermond et al. [18] provided a summary of the stability conditions for different projection schemes by reducing the analysis of the unsteady incompressible Navier-Stokes equations to the equivalent steady Stokes problem. This same approach will be considered in following sections. A particularly interesting conclusion from [18] is that the stability condition for the Chorin-Temam scheme to be stable depends on the time step $\triangle t$. Namely, the time step is required to be large enough for the scheme to remain stable. Observations are reported for the time step limit for stability $\triangle t_{\text{lim}}$ when considering low order (e.g. linear finite element formulations, relying only on h-refinement) and high order spectral type methods (e.g. Fourier, Chebyshev, allowing p-refinement):

- $\triangle t_{\text{lim}} \geq c_1 h^2$ for low order finite element methods, where c_1 is a constant independent of the spatial discretisation defined through the characteristic mesh size h,
- $\triangle t_{\text{lim}} \geq c_2 k^{-3}$ for high order spectral discretisations, where c_2 is a constant independent of the spatial discretisation defined through the polynomial order k.

In this work, we examine these observations and provide an explanation for such behaviour. Without loss of generality, we focus on the analysis of the popular velocity-correction scheme proposed by Orszag et al. [29], Karniadakis et al. [24] and theoretically analysed by Guermond et al. [18], which under the steady Stokes assumption can be shown to resemble the Chorin-Temam scheme. In particular, we will study the effects of using such projection schemes together with high order spatial discretisations,