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Fourth Order Difference Approximations for Space Riemann-Liouville Derivatives Based on Weighted and Shifted Lubich Difference Operators

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Abstract. High order discretization schemes play more important role in fractional operators than classical ones. This is because usually for classical derivatives the stencil for high order discretization schemes is wider than low order ones; but for fractional operators the stencils for high order schemes and low order ones are the same. Then using high order schemes to solve fractional equations leads to almost the same computational cost with first order schemes but the accuracy is greatly improved. Using the fractional linear multistep methods, Lubich obtains the ν-th order ($\nu \leq 6$) approximations of the α-th derivative ($\alpha > 0$) or integral ($\alpha < 0$) [Lubich, SIAM J. Math. Anal., 17, 704-719, 1986], because of the stability issue the obtained scheme can not be directly applied to the space fractional operator with $\alpha \in (1,2)$ for time dependent problem. By weighting and shifting Lubich's 2nd order discretization scheme, in [Chen & Deng, SINUM, arXiv:1304.7425] we derive a series of effective high order discretizations for space fractional derivative, called WSLD operators there. As the sequel of the previous work, we further provide new high order schemes for space fractional derivatives by weighting and shifting Lubich's 3rd and 4th order discretizations. In particular, we prove that the obtained 4th order approximations are effective for space fractional derivatives. And the corresponding schemes are used to solve the space fractional diffusion equation with variable coefficients.

AMS subject classifications: 26A33, 03F35, 47B37, 65M12

Key words: Fractional derivatives, high order scheme, weighted and shifted Lubich difference (WSLD) operators, numerical stability.

1 Introduction

Fractional calculus (i.e., integrals and derivatives of any arbitrary real or even complex order) has attracted considerable attention during the past several decades, due mainly

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to its demonstrated applications in seemingly diverse and widespread fields of science and engineering [11]; and fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes [17]. With the ubiquitous applications of fractional calculus, fractional partial differential equations (PDEs) appear naturally. Effectively solving fractional PDEs becomes urgent, and intrigues mathematicians. It is still possible to analytically solve the linear fractional PDEs with constant coefficients by using Laplace or Fourier transform, but most of the time the solutions are represented by infinite series or transactional functions. Without doubt, the new challenges also exist in numerically solving fractional PDEs; but some basic ideas have been developed, for instance, finite difference method [15, 20–22, 25]; finite element method [7, 8, 23]; spectral method [12, 13].

In numerically solving fractional PDEs, besides a little bit complex numerical analysis, the big challenge comes from the computational cost caused by the nonlocal properties of fractional operators. High order scheme is a natural idea to reduce the challenge of cost. Comparing with first order schemes, the high order schemes for fractional operators do not increase computational cost but greatly improve the accuracy. The reason is that both the derived matrixes corresponding to the higher order schemes and low order schemes are full and have the same structure [3]. In fact, there are already some important progresses for the high order discretizations of fractional derivatives, including WSGD operator [22], CWSGD operator [24], second order discretization [4, 5, 20], second order discretization for Riesz fractional derivative [16], and WSLD operator [2]. This paper is the sequel of [2], i.e, based on Lubich's 3rd and 4th operators to provide new high order discretization schemes for space fractional derivatives.

Using the fractional linear multistep methods, [14] obtains the ν-th order ($\nu \le 6$) approximations of the α-th derivative ($\alpha > 0$) or integral ($\alpha < 0$) by the corresponding coefficients of the generating functions $\delta^\alpha(\zeta)$, where

$$\delta^\alpha(\zeta) = \left(\sum_{i=1}^{\nu} \frac{1}{i} (1-\zeta)^i \right)^\alpha. \tag{1.1}$$

For $\alpha = 1$, the scheme reduces to the classical $(\nu+1)$-point backward difference formula [10]. For $\nu = 1$, $\alpha > 0$, the scheme (1.1) corresponds to the standard Grünwald discretization of α-th derivative with first order accuracy; unfortunately, for the time dependent equations all the difference discretizations are unstable. By weighting and shifting Lubich's 2nd order discretization, a class of effective high order schemes for space fractional derivatives are presented [2]. Is it possible to design the high order schemes for space fractional derivatives by using Lubich's 3rd, 4th, 5th, 6th order operators? This paper will answer that at least by applying Lubich's 3rd, 4th order operators, the new discretizations for space fractional derivatives can be constructed. The concrete discretizations will be presented, and the effectiveness of 4th order schemes for space fractional derivative will be proved. And we will also provide a simple application to solve the space fractional diffusion equation with variable coefficients.