Commun. Comput. Phys.
doi: 10.4208/cicp.221212.300114a

Vol. **16**, No. 2, pp. 348-364
August 2014

Numerical Study of Singularity Formation in Relativistic Euler Flows

Pierre A. Gremaud[1] and Yi Sun[2,*]

[1] *Department of Mathematics, North Carolina State University, Raleigh, NC, 27695, USA.*
[2] *Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.*

Received 22 December 2012; Accepted (in revised version) 30 January 2014

Communicated by Jan S. Hesthaven

Available online 17 April 2014

Abstract. The formation of singularities in relativistic flows is not well understood. Smooth solutions to the relativistic Euler equations are known to have a finite life-span; the possible breakdown mechanisms are shock formation, violation of the sub-luminal conditions and mass concentration. We propose a new hybrid Glimm/central-upwind scheme for relativistic flows. The scheme is used to numerically investigate, for a family of problems, which of the above mechanisms is involved.

AMS subject classifications: 35L67, 65M08, 65M25, 76Y05, 83C05, 83C75

Key words: Relativistic Euler equations, singularity formation, Glimm scheme, central-upwind scheme, hybrid method.

1 Introduction

Relativistic hydrodynamics plays a fundamental role in many fields of physics from astrophysics and cosmology to nuclear physics [3,38]. The relativistic Euler equations considered in this paper describe the dynamics of a compressible perfect fluid in the context of special relativity, i.e., the fluid evolves in a flat Minkowski spacetime. These equations are valid away from large matter concentrations and in small regions of spacetime. They can be regarded as an approximation to the Euler-Einstein equations.

Elementary nonlinear waves for the relativistic Euler equations have been analyzed [1,25,35,36]. Existence of entropy solutions for problems without vacuum state [33] as well as for problems with possible vacuum formation [22] has been established. The

*Corresponding author. *Email addresses:* `gremaud@ncsu.edu` (P. A. Gremaud), `yisun@math.sc.edu` (Y. Sun)

mechanism by which singularities form is, however, not fully understood. While Pan and Smoller [29] has shown finite time singularity formation of any smooth solutions of the relativistic Euler equations for a perfect fluid, see Section 2, the *type* of singularity which occurs is unknown.

We present here a numerical investigation of singularity formation for solutions of the relativistic Euler equations both in two and three spatial dimensions with radially, respectively spherically, symmetric smooth initial data. We review existing numerical methods for relativistic hydrodynamics in Section 3. In order to numerically characterize the mechanisms by which singularities form, we propose a hybrid approach that combines standard finite difference methods in the smooth part of the flow with Glimm scheme near discontinuities and sharp gradients. The detection of sharp gradient areas is discussed in Section 3; details about the Riemann solver for the Glimm scheme are discussed in the Appendix.

The character of the numerical solutions is analyzed in a post-processing step. Naive ideas such as checking directly whether the Rankine-Hugoniot jump condition [23,32,37] holds at discontinuities are highly impractical because of the difficulties in evaluating all involved quantities. Instead, we use a detailed study of the numerical characteristic curves which is described in Section 4, along with our numerical results. For a family of problems in (2+1) dimensional spacetime with radial symmetry and (3+1) dimensional spacetime with spherical symmetry, we show that singularities occur through shock formation.

2 Basic equations and mathematical analysis

The relativistic Euler equations for a perfect fluid can be written as [29,33]

$$\partial_t \left(\frac{\rho c^2 + p}{c^2 - |v|^2} - \frac{p}{c^2} \right) + \nabla_x \cdot \left(\frac{\rho c^2 + p}{c^2 - |v|^2} v \right) = 0, \tag{2.1}$$

$$\partial_t \left(\frac{\rho c^2 + p}{c^2 - |v|^2} v \right) + \nabla_x \cdot \left(\frac{\rho c^2 + p}{c^2 - |v|^2} v \otimes v \right) + \nabla_x p = 0, \tag{2.2}$$

where ρ is the mass-energy density, p the pressure, c the speed of light. The vector v is defined as

$$v = \frac{cu}{\sqrt{1+|u|^2}},$$

where u is the velocity of the fluid. In d space dimensions ($d=2$ or 3), u, v and the space coordinates x are d-vectors. Hereafter, for convenience, we refer to v as the velocity. The system (2.1)-(2.2) is closed by an equation of state which we take as the γ-law:

$$p(\rho) = \sigma^2 \rho^\gamma, \qquad \gamma \geq 1. \tag{2.3}$$

Further, a subluminal condition is assumed, i.e.

$$0 < p'(\rho) < c^2, \qquad \text{for } \rho \in (\rho_1, \rho_2), \tag{2.4}$$