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IMEX Large Time Step Finite Volume Methods for Low Froude Number Shallow Water Flows

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Abstract. We present new large time step methods for the shallow water flows in the low Froude number limit. In order to take into account multiscale phenomena that typically appear in geophysical flows nonlinear fluxes are split into a linear part governing the gravitational waves and the nonlinear advection. We propose to approximate fast linear waves implicitly in time and in space by means of a genuinely multidimensional evolution operator. On the other hand, we approximate nonlinear advection part explicitly in time and in space by means of the method of characteristics or some standard numerical flux function. Time integration is realized by the implicit-explicit (IMEX) method. We apply the IMEX Euler scheme, two step Runge Kutta Cranck Nicolson scheme, as well as the semi-implicit BDF scheme and prove their asymptotic preserving property in the low Froude number limit. Numerical experiments demonstrate stability, accuracy and robustness of these new large time step finite volume schemes with respect to small Froude number.

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1 Introduction

In oceanography, meteorology or river flow engineering shallow water models are used to describe a thin layer of constant density fluid in hydrostatic balance bounded from

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below by a rigid surface, see, e.g., [8,11,22,45]. The shallow water equations (SWE)

$$\begin{bmatrix} h \\ hu \\ hv \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2+\frac{1}{2\varepsilon^2}h^2 \\ huv \end{bmatrix}_x + \begin{bmatrix} hv \\ huv \\ hv^2+\frac{1}{2\varepsilon^2}h^2 \end{bmatrix}_y = \begin{bmatrix} 0 \\ -\frac{1}{\varepsilon^2}h\tilde{b}_x \\ -\frac{1}{\varepsilon^2}h\tilde{b}_y \end{bmatrix} \tag{1.1}$$

describe the motion of shallow water, where h is the water depth, $\mathbf{u}=(u,v)$; u,v are the velocities in x- and y-direction and $\tilde{b}$ is time independent bottom topography. Further, $\varepsilon := u_{ref}/c_{ref} = u_{ref}/\sqrt{gh_{ref}}$ is the reference Froude number, g is the gravitational constant, u_{ref} and h_{ref} are the problem dependent reference values for velocity and water depth, respectively. System (1.1) is a hyperbolic balance law, which can be derived by integrating the Navier-Stokes equations along the vertical axis [45].

Let us note that geophysical flows are typically perturbations of some underlying equilibrium state. One possibility to take the loss of significance into account is to approximate just the perturbation of the equilibrium states [15,34]. For the shallow water equations (1.1) the so-called lake at rest solution $h+b=\text{const.}$, $u=0=v$ is the equilibrium state.

We would like to point out, that in literature there are already several approaches that describe how to design a numerical scheme which satisfies some important equilibrium conditions, such as the lake at rest state or the geostrophic equilibrium, exactly for given discrete data. Such schemes are called *well-balanced schemes* or schemes satisfying the so-called *C-property*, we refer a reader to, e.g., [4,7,10,16,24–27,32] and to [5], where the C-property has been introduced firstly. We will discuss the question of well-balancing more deeply in Sections 4.3 and 5 and show that our newly developed large time step schemes are well-balanced for the lake at rest uniformly with respect to the Froude number ε.

Now, we introduce the following variable transformation $w=(z,m,n):=(h+b,hu,hv)$. Here z is the perturbation of the constant water level $H=h+\tilde{b}$ and $b=\tilde{b}-RBC<0$ with a problem defined relative bottom topography constant RBC. We should also note that an analogous variable transformation has been already used in [24,26,27,41,42]. The only difference in our case is that we introduce explicitly a "shift" of the coordinate system in the vertical direction by a suitable constant denoted by RBC in order to obtain a still water level to be zero. Consequently, we aim to have the perturbation z to be a small positive or negative value. Note that by this transformation we obtain bottom topography function $b<0$. System (1.1) can be now rewritten in the non-dimensional form using the new variables z,m,n

$$\begin{bmatrix} z \\ m \\ n \end{bmatrix}_t + \begin{bmatrix} m \\ \frac{m^2}{z-b}+\frac{1}{2\varepsilon^2}z^2-\frac{1}{\varepsilon^2}zb \\ \frac{mn}{z-b} \end{bmatrix}_x + \begin{bmatrix} n \\ \frac{mn}{z-b} \\ \frac{n^2}{z-b}+\frac{1}{2\varepsilon^2}z^2-\frac{1}{\varepsilon^2}zb \end{bmatrix}_y = \begin{bmatrix} 0 \\ -\frac{1}{\varepsilon^2}zb_x \\ -\frac{1}{\varepsilon^2}zb_y \end{bmatrix}. \tag{1.2}$$

In geophysical problems *low Froude number shallow water flows* typically appear, cf. [22,33,45]. This means that the advection speed u_{ref} is much smaller then the speed