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A Numerical Method and its Error Estimates for the Decoupled Forward-Backward Stochastic Differential Equations

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Abstract. In this paper, a new numerical method for solving the decoupled forward-backward stochastic differential equations (FBSDEs) is proposed based on some specially derived reference equations. We rigorously analyze errors of the proposed method under general situations. Then we present error estimates for each of the specific cases when some classical numerical schemes for solving the forward SDE are taken in the method; in particular, we prove that the proposed method is second-order accurate if used together with the order-2.0 weak Taylor scheme for the SDE. Some examples are also given to numerically demonstrate the accuracy of the proposed method and verify the theoretical results.

AMS subject classifications: 60H35, 60H10, 65C20, 65C30

Key words: Decoupled forward-backward stochastic differential equations, numerical scheme, error estimates.

1 Introduction

Let $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{0\le t\le T})$ be a complete, filtered probability space on which a standard d-dimensional Brownian motion $W_t=(W_t^1,W_t^2,\cdots,W_t^d)^*$ is defined, such that $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration of the Brownian motion W_t and all the P-null sets are augmented to each σ-field $\mathcal{F}_t$. Here the operator $(\cdot)^*$ denotes the transpose operator for a matrix or

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vector. We consider the decoupled forward-backward stochastic differential equations (FBSDEs) on $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{0\le t\le T})$

$$\begin{cases} X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s, & t\in[0,T], \\ Y_t = \varphi(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, & t\in[0,T], \end{cases} \tag{1.1}$$

with the functions $b(t,X)$: $[0,T]\times\mathbb{R}^q\to\mathbb{R}^q$, $\sigma(t,X)$: $[0,T]\times\mathbb{R}^q\to\mathbb{R}^{q\times d}$, $f(t,X,Y,Z)$: $[0,T]\times\mathbb{R}^q\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\to\mathbb{R}^m$, and $\varphi(\cdot)$: $\mathbb{R}^q\to\mathbb{R}^m$. Note that the integrals in (1.1) with respect to the d-dimensional Brownian motion W_s are the Itô type stochastic integrals. The first equation in (1.1) is the standard (forward) stochastic differential equation (SDE), and the second equation is the so-called backward stochastic differential equation (BSDE). A process (X_t,Y_t,Z_t) is called an L^2 solution of the decoupled FBSDEs (1.1) if it is $\{\mathcal{F}_t\}$-adapted and square integrable and satisfy (1.1). In the sequel, a solution means a L^2 solution. Under standard conditions on f and φ, Pardoux and Peng [25] originally proved the existence and uniqueness of solution of nonlinear BSDEs. Since then a lot of efforts have been devoted to study of FBSDEs [2–6, 8–11, 13, 14, 19–24, 30, 32] due to their natural applications in many fields including mathematical finance, partial differential equations (PDEs), stochastic PDEs, stochastic control, risk measure, game theory, and so on.

It is well-known that it is often difficult to obtain analytic solutions in the close form for the FBSDEs, even for the linear case, so that computing approximate solutions of FBSDEs becomes highly desired. There are lots of works on numerical methods for numerically solving BSDEs. Based on the relation between the FBSDEs and their corresponding parabolic partial differential equations (PDEs) [7,10,12,17,26], some algorithms were proposed to solve FBSDEs in [5, 10, 11, 21–24]. There are also some other numerical methods for solving BSDEs or FBSDEs, which were proposed based on directly discretizing BSDEs or FBSDEs [2,4,6,8,13,28,29,33–35]. Many existing numerical methods for the decoupled FBSDEs (1.1) are half order and one order in time such as those in [3, 4, 8, 10, 13, 14, 20, 29]. In these methods, forward or backward trapezoidal rules were often used to approximate the integrals in (1.1), and the martingale representation was used in their error analysis. In this paper, based on properties of the Itô's integral and the nature of solution of the FBSDEs, we will propose a numerical method for solving the decoupled FBSDEs (1.1) that utilizes the trapezoidal rule and approximations of some reference equations with a newly defined standard Brownian motion. We rigorously derive error estimates for this method for general cases. Under certain regularity assumptions on the functions b, σ, f and φ, we also show that the proposed scheme can be up to second-order accurate in time.

Now let us introduce some notations which will be used in this paper:

- $|\cdot|$: the standard Euclidean norm in the Euclidean space $\mathbb{R}$, $\mathbb{R}^q$ and $\mathbb{R}^{q\times d}$.
- $L^2 = L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$: the set of all $\mathcal{F}_t$-adapted and mean-square-integrable processes valued in $\mathbb{R}^d$.