Commun. Comput. Phys.
doi: 10.4208/cicp.060712.210313a

Vol. **15**, No. 1, pp. 46-75
January 2014

A Godunov-Type Solver for the Numerical Approximation of Gravitational Flows

J. Vides[1,*], B. Braconnier[2], E. Audit[3], C. Berthon[4] and B. Nkonga[5,6]

[1] *Inria, Maison de la Simulation, USR 3441, Gif-sur-Yvette, France.*
[2] *IFP Energies Nouvelles, 1-4 avenue de Bois-Préau, 92852 Rueil-Malmaison, France.*
[3] *CEA, Maison de la Simulation, USR 3441, Gif-sur-Yvette, France.*
[4] *Univ. of Nantes, Lab. J. Leray, UMR CNRS 6629, Nantes, France.*
[5] *Univ. of Nice-Sophia Antipolis, Lab. J.A. Dieudonné, UMR CNRS 7351, Nice, France.*
[6] *Inria Sophia Antipolis, BP. 93, F-06902 Sophia Antipolis Cedex, France.*

Received 6 July 2012; Accepted (in revised version) 21 March 2013

Communicated by Chi-Wang Shu

Available online 26 July 2013

Abstract. We present a new numerical method to approximate the solutions of an Euler-Poisson model, which is inherent to astrophysical flows where gravity plays an important role. We propose a discretization of gravity which ensures adequate coupling of the Poisson and Euler equations, paying particular attention to the gravity source term involved in the latter equations. In order to approximate this source term, its discretization is introduced into the approximate Riemann solver used for the Euler equations. A relaxation scheme is involved and its robustness is established. The method has been implemented in the software HERACLES [29] and several numerical experiments involving gravitational flows for astrophysics highlight the scheme.

AMS subject classifications: 85-08, 65Z05, 35L65, 35L45, 76M12

Key words: Euler-Poisson equations, approximate Riemann solver, relaxation scheme, source terms, gravitational effects.

1 Introduction

The present paper is devoted to the numerical approximation of the Euler equations when gravitational effects are taken into account. The associated solutions are governed

*Corresponding author. *Email addresses:* jeaniffer-lissette.vides_higueros@inria.fr (J. Vides), benjamin.braconnier@ifpenergiesnouvelles.fr (B. Braconnier), edouard.audit@cea.fr (E. Audit), christophe.berthon@math.univ-nantes.fr (C. Berthon), boniface.nkonga@unice.fr (B. Nkonga)

by an Euler-Poisson model, given by the following system of partial differential equations (PDEs):

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}+p) = -\rho\nabla\phi, \\ \partial_t(\rho E) + \nabla\cdot((\rho E+p)\mathbf{u}) = -\rho\mathbf{u}\cdot\nabla\phi, \\ \Delta\phi = 4\pi G\rho, \end{cases} \tag{1.1}$$

where $\rho > 0$ is the density, $\mathbf{u}\in\mathbb{R}^d$ the velocity, and E the specific energy. The integer d refers to the space dimension. The thermodynamic pressure p is assumed to be governed by an equation of state

$$p := p(\rho,\epsilon),$$

where $\epsilon = E - |\mathbf{u}|^2/2$ represents the specific internal energy. As usual (see [28, 49]), the pressure law is assumed to satisfy

$$\partial_\rho p(\rho,\epsilon) + \frac{p(\rho,\epsilon)}{\rho^2}\partial_\epsilon p(\rho,\epsilon) > 0.$$

Here, $G \approx 6.67\times 10^{-11} m^3 kg^{-1} s^{-2}$ is the gravitational constant. The gravitational potential ϕ is always a smooth function since it is the solution of the Laplace equation. For the sake of notation simplicity, the system is rewritten in compact form as

$$\begin{cases} \partial_t \mathbf{W} + \nabla\cdot\mathbf{F}(\mathbf{W}) + \mathbf{B}(\mathbf{W})\nabla\phi = 0, \\ \Delta\phi = 4\pi G\rho, \end{cases} \tag{1.2}$$

with

$$\mathbf{W} = (\rho, \rho\mathbf{u}, \rho E)^T, \tag{1.3a}$$

$$\mathbf{F}(\mathbf{W}) = (\rho\mathbf{u}, \rho\mathbf{u}\otimes\mathbf{u}+p, (\rho E+p)\mathbf{u})^T, \tag{1.3b}$$

$$\mathbf{B}(\mathbf{W}) = \rho\begin{pmatrix} \mathbf{0}_d^T \\ \mathcal{I}_d \\ \mathbf{u}^T \end{pmatrix}, \tag{1.3c}$$

where $\mathbf{0}_d = (0,\cdots,0)^T$ is the null vector in $\mathbb{R}^d$ and $\mathcal{I}_d$ is an identity matrix of dimension d. As usual, $\mathbf{W}:\mathbb{R}^d\times\mathbb{R}^+\to\Omega_d$ is the state vector and $\mathbf{F}:\Omega_d\to\mathbb{R}^{2+d}$ is the flux function. Here, $\mathbf{B}:\Omega_d\to\mathbb{R}^{2+d}\times\mathbb{R}^d$ represents the gravitational contribution when multiplied by $\nabla\phi$. The convex set Ω_d of the admissible state vectors is defined by

$$\Omega_d = \left\{\mathbf{W}\in\mathbb{R}^{2+d};\, \rho>0,\, \mathbf{u}\in\mathbb{R}^d,\, \epsilon = E - \frac{|\mathbf{u}|^2}{2} > 0\right\}. \tag{1.4}$$

System (1.2) is completed with appropriate initial and boundary conditions that depend on the problem being considered, as will be seen in Section 5. Nevertheless, we