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Numerical Study of Quantized Vortex Interaction in the Ginzburg-Landau Equation on Bounded Domains

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Abstract. In this paper, we study numerically quantized vortex dynamics and their interaction in the two-dimensional (2D) Ginzburg-Landau equation (GLE) with a dimensionless parameter $\varepsilon > 0$ on bounded domains under either Dirichlet or homogeneous Neumann boundary condition. We begin with a review of the reduced dynamical laws for time evolution of quantized vortex centers in GLE and show how to solve these nonlinear ordinary differential equations numerically. Then we present efficient and accurate numerical methods for discretizing the GLE on either a rectangular or a disk domain under either Dirichlet or homogeneous Neumann boundary condition. Based on these efficient and accurate numerical methods for GLE and the reduced dynamical laws, we simulate quantized vortex interaction of GLE with different ε and under different initial setups including single vortex, vortex pair, vortex dipole and vortex lattice, compare them with those obtained from the corresponding reduced dynamical laws, and identify the cases where the reduced dynamical laws agree qualitatively and/or quantitatively as well as fail to agree with those from GLE on vortex interaction. Finally, we also obtain numerically different patterns of the steady states for quantized vortex lattices under the GLE dynamics on bounded domains.

AMS subject classifications: 35Q56, 65M06, 65M70, 82D55

Key words: Ginzburg-Landau equation, quantized vortex, Dirichlet boundary condition, homogeneous Neumann boundary condition, reduced dynamical laws, time splitting, compact finite difference method, finite element method.

1 Introduction

A quantized vortex in two-dimensional (2D) space is a particle-like or topological defect, whose center is zero of the order parameter, possessing localized phase singularity with

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integer topological charge called also as winding number or index. Quantized vortices have been widely observed in many different physical systems, such as the liquid helium, type-II superconductors, Bose-Einstein condensates, atomic gases and nonlinear optics [1, 5, 10, 12, 23, 28]. They are key signatures of superconductivity and superfluidity and their study retains as one of the most important and fundamental problems since they were predicted by Lars Onsager in 1947 in connection with superfluid Helium.

In this paper, we are concerned with the vortex dynamics and interactions in 2D Ginzburg-Landau equation (GLE) for modelling superconductivity [15, 23, 25]:

$$\lambda_\varepsilon \partial_t \psi^\varepsilon(\mathbf{x},t) = \Delta\psi^\varepsilon + \frac{1}{\varepsilon^2}(1-|\psi^\varepsilon|^2)\psi^\varepsilon, \qquad \mathbf{x}\in\Omega, \quad t>0, \tag{1.1}$$

with initial condition

$$\psi^\varepsilon(\mathbf{x},0) = \psi_0^\varepsilon(\mathbf{x}), \qquad \mathbf{x}\in\bar{\Omega}, \tag{1.2}$$

and under either Dirichlet boundary condition (BC)

$$\psi^\varepsilon(\mathbf{x},t) = g(\mathbf{x}) = e^{i\omega(\mathbf{x})}, \qquad \mathbf{x}\in\partial\Omega, \quad t\geq 0, \tag{1.3}$$

or homogeneous Neumann BC

$$\frac{\partial\psi^\varepsilon(\mathbf{x},t)}{\partial\mathbf{n}} = 0, \qquad \mathbf{x}\in\partial\Omega, \quad t\geq 0. \tag{1.4}$$

Here, $\Omega\subset\mathbb{R}^2$ is a 2D smooth and bounded domain, t is time, $\mathbf{x}=(x,y)\in\mathbb{R}^2$ is the Cartesian coordinate vector, $\psi^\varepsilon := \psi^\varepsilon(\mathbf{x},t)$ is a complex-valued function describing the 'order parameter' for a superconductor, ω is a given real-valued function, ψ_0^ε and g are given smooth and complex-valued functions satisfying the compatibility condition $\psi_0^\varepsilon(\mathbf{x})=g(\mathbf{x})$ for $\mathbf{x}\in\partial\Omega$, $\mathbf{n}=(n_1,n_2)$ and $\mathbf{n}_\perp=(-n_2,n_1)\in\mathbb{R}^2$ satisfying $|\mathbf{n}|=\sqrt{n_1^2+n_2^2}=1$ are the outward normal and tangent vectors along $\partial\Omega$, respectively, $\varepsilon>0$ is a given dimensionless constant, and λ_ε is a positive function of ε. Denote the Ginzburg-Landau (GL) functional ('energy') as [15, 23, 25]

$$\mathcal{E}^\varepsilon(t) := \int_\Omega \left[\frac{1}{2}|\nabla\psi^\varepsilon|^2 + \frac{1}{4\varepsilon^2}\left(|\psi^\varepsilon|^2-1\right)^2\right] d\mathbf{x} = \mathcal{E}_{\text{kin}}^\varepsilon(t) + \mathcal{E}_{\text{int}}^\varepsilon(t), \qquad t\geq 0, \tag{1.5}$$

where the kinetic and interaction parts are defined as

$$\mathcal{E}_{\text{kin}}^\varepsilon(t) := \frac{1}{2}\int_\Omega |\nabla\psi^\varepsilon|^2 d\mathbf{x}, \quad \mathcal{E}_{\text{int}}^\varepsilon(t) := \frac{1}{4\varepsilon^2}\int_\Omega \left(|\psi^\varepsilon|^2-1\right)^2 d\mathbf{x}, \qquad t\geq 0,$$

then it is easy to see that, for GLE (1.1) with either Dirichlet BC (1.3) or homogeneous Neumann BC (1.4) for general domain Ω, or periodic BC when Ω is a rectangle, the GL functional decreases when time increases, i.e.

$$\frac{d}{dt}\mathcal{E}^\varepsilon(t) = -\lambda_\varepsilon \int_\Omega |\partial_t\psi^\varepsilon|^2 d\mathbf{x} \leq 0, \qquad t\geq 0.$$