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# Degenerate Anisotropic Elliptic Problems and Magnetized Plasma Simulations

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**Abstract.** This paper is devoted to the numerical approximation of a degenerate anisotropic elliptic problem. The numerical method is designed for arbitrary space-dependent anisotropy directions and does not require any specially adapted coordinate system. It is also designed to be equally accurate in the strongly and the mildly anisotropic cases. The method is applied to the Euler-Lorentz system, in the drift-fluid limit. This system provides a model for magnetized plasmas.

**AMS subject classifications**: 65N06, 65N12, 65M06, 65M12, 76W05, 76X05, 76N17

**Key words**: Anisotropic elliptic problem, asymptotic-preserving scheme, Lorentz force, large magnetic field, low-Mach number, drift-fluid limit.

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## 1 Introduction

This paper discusses the numerical resolution of degenerate anisotropic elliptic problems of the form:

$$-(b\cdot\nabla)\big(\nabla\cdot(b\phi^\varepsilon)\big)+\varepsilon\phi^\varepsilon=f^\varepsilon, \qquad \text{in } \Omega, \tag{1.1a}$$

$$(b\cdot\nu)\nabla\cdot\big(b\phi^\varepsilon\big)=0, \qquad \text{on } \partial\Omega, \tag{1.1b}$$

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where $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, $f^\varepsilon$ is a given function, $b$ is a normalized vector field defining the anisotropy direction and $\varepsilon$ measures the strength of this anisotropy. In this expression $\nabla$ and $\nabla\cdot$ are respectively the gradient and divergence operators. The unit outward normal at $x\in\partial\Omega$ is denoted by $\nu$. In the context of plasmas, $\varepsilon$ is related to the gyro period (i.e., the period of the gyration motion of the particles about the magnetic field lines) and the anisotropy direction $b$ satisfies $b=B/|B|$ with the magnetic field $B$ verifying $\nabla\cdot B=0$. Eq. (1.1a) may also arise in other contexts, such as rapidly rotating flows, shell theory and may also be found when special types of semi-implicit time discretization of diffusion equations are used.

The elliptic equation is not in the usual divergence form due to an exchange between the gradient and divergence operators. However, the methodology would apply equally well to the operator $\nabla\cdot((b\otimes b)\cdot\nabla\phi))$, up to some simple changes. The expression considered here is motivated by the application to the Euler-Lorentz system of plasmas. This application has already been considered in a previous study [13] but we introduce two important developments. First the present numerical method does not request the development of a special coordinate system adapted to $b$. In [13], $b$ was assumed aligned with one coordinate direction. Second, the present paper considers Neumann boundary conditions instead of Dirichlet ones as in [13]. Although seemingly innocuous, this change brings in a considerable difficulty, linked with the degeneracy of the limit problem, as explained below.

A classical discretization of problem (1.1a), (1.1b) leads to an ill-conditioned linear system as $\varepsilon\to 0$. Indeed setting formally $\varepsilon=0$ in (1.1a), (1.1b), we get:

$$-(b\cdot\nabla)\nabla\cdot(b\psi)=f^{(0)}, \qquad \text{in } \Omega, \tag{1.2a}$$

$$(b\cdot\nu)\nabla\cdot(b\psi)=0, \qquad \text{on } \partial\Omega, \tag{1.2b}$$

with $f^{(0)}=\lim_{\varepsilon\to 0}f^\varepsilon$. The homogeneous system associated to (1.2a), (1.2b) admits an infinite number of solutions, namely all functions $\psi$ satisfying $\nabla\cdot(b\psi)=0$. This degeneracy results from the Neumann boundary conditions (1.2b) and would also occur if periodic boundary conditions were used. On the other hand, (1.2a) is not degenerate if supplemented with Dirichlet or Robin conditions, which was the case considered in [13]. A standard numerical approximation of (1.2a), (1.2b) generates a matrix whose condition number blows up as $\varepsilon\to 0$, leading to very time consuming and/or poorly accurate solution algorithms.

To bypass these limitations, we follow the idea introduced in [12] and use a decomposition of the solution in its average along the $b$-field lines and a fluctuation about this average. This decomposition ensures an accurate computation of the solution for all values of $\varepsilon$. In [12], this decomposition approach was developed for a uniform $b$ and a coordinate system with one coordinate direction aligned with $b$. To extend this approach to arbitrary anisotropy fields $b$, a possible way is to use an adapted curvilinear coordinate system with one coordinate curve tangent to $b$. This is the route followed by [4], which proposes an extension of [12] in the context of ionospheric plasma physics, where the