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Adaptive Stroud Stochastic Collocation Method for Flow in Random Porous Media via Karhunen-Loève Expansion

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Abstract. In this paper we develop a Stochastic Collocation Method (SCM) for flow in randomly heterogeneous porous media. At first, the Karhunen-Loève expansion is taken to decompose the log transformed hydraulic conductivity field, which leads to a stochastic PDE that only depends on a finite number of i.i.d. Gaussian random variables. Based on the eigenvalue decay property and a rough error estimate of Stroud cubature in SCM, we propose to subdivide the leading dimensions in the integration space for random variables to increase the accuracy. We refer to this approach as *adaptive Stroud SCM*. One- and two-dimensional steady-state single phase flow examples are simulated with the new method, and comparisons are made with other stochastic methods, namely, the Monte Carlo method, the tensor product SCM, and the quasi-Monte Carlo SCM. The results indicate that the adaptive Stroud SCM is more efficient and the statistical moments of the hydraulic head can be more accurately estimated.

AMS subject classifications: 60H15, 65M70, 76M22, 76S05

Key words: Adaptive Stroud stochastic collocation method, Karhunen-Loève expansion, Monte Carlo simulation, random porous flow.

1 Introduction

Geological formation material properties are ordinarily observed at a few locations although they exhibit a high degree of spatial variability. This leads to uncertainty about

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the prediction of subsurface flow. In order to quantify the uncertainty, a stochastic description of the medium properties is needed. This brings the traditional porous medium equations into stochastic partial differential equations (SPDE) [8, 23, 37], which possess more interesting and challenging computational problems.

Monte Carlo (MC) simulation is one of the most natural approaches to solve SPDE. It is a statistical method and the basic idea is to sample a large amount of realizations for the random process and approximate the moments of interest with ensemble average. Thus the number of realizations, which one chooses, controls the accuracy of MC simulation. To ensure the convergence of the moments, typically a few thousand samples or more are required, which is the main disadvantage of the direct sampling MC simulation.

An alternative approach is based on moment equations [10, 13, 20, 37]. This method usually leads to a system of deterministic differential equations which govern the propagation of the statistical moments of the random variables (fields). For deriving these equations, the method of perturbation or some type of closure approximation is needed. However, the computational effort is still high. To compute the hydraulic head covariance to first order in the variability of the log hydraulic conductivity, one needs to solve the deterministic equations for the cross-covariance between the hydraulic head and the log hydraulic conductivity and those for the hydraulic head covariance. If higher-order terms are included, the computational effort will increase dramatically. Zhang et al. recently developed a Karhunen-Loève expansion based moment equation approach (KLME) and applied this method to flows in porous media [5, 6, 17, 35, 38]. With this approach, the equations for the coefficients are uncoupled. One can obtain the high-order terms of the mean and variance of hydraulic head with relatively small computational efforts. The approach can be easily implemented with existing simulators. By these advantages, the KLME approach is generally more efficient than the traditional moment equation approach.

A mathematically unified numerical approach for SPDE — Stochastic Finite Element Method (SFEM) — is also under study and has been rapidly developed in recent years [1, 2, 9, 12, 15]. This method employs the polynomial chaos expansion (PCE) for random processes. After truncation in probability space, its formulation fits into the traditional spectral methods framework, which ensures exponential convergence in probability space [1, 12, 31–33]. However, as the deterministic spectral methods, one must solve a set of coupled equations for the deterministic coefficients of the PCE. This increases the computational effort when the number of coefficients is large. In the original form of PCE, it is based on the Hermite polynomial expansions in terms of Gaussian random variables. Xiu et al. generalized the formulation into Wiener-Askey polynomial basis for other types of random variables, which they called generalized polynomial chaos expansion (gPC) [31–33].

To overcome the difficulty for solving the coupled system, the stochastic collocation method (SCM) was first proposed by Tatang et al. [26]. It is successfully applied and made more practical in [18, 29, 30, 34]. In this approach, the random variables are represented by Lagrange interpolation polynomials and one can derive an uncoupled system