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Comparison Principles for Some Fully Nonlinear Sub-Elliptic Equations on the Heisenberg Group

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Abstract. In this paper, we prove a form of the strong comparison principle for a class of fully nonlinear subelliptic operators of the form $\nabla^2_{H,s}\psi+L(\cdot,\psi,\nabla_H\psi)$ on the Heisenberg group, which include the CR invariant operators.

Key Words: Comparison principle, subellipticity, CR invariance, Heisenberg group, propagation of touching points.

AMS Subject Classifications: 35J60, 35J70, 35B51, 35B65, 35D40, 53C21, 58J70

1 Introduction

In this paper, we establish a form of the comparison principle for a class of subelliptic equations on the Heisenberg group.

Let Ω be an open connected subset of $\mathbb{R}^n$ $(n\geq 1)$, the n-dimensional Euclidean space. Assume that u, $v\in C^2(\Omega)$ satisfy

$$u\geq v \quad \text{in} \quad \Omega. \tag{1.1}$$

The standard form of the strong comparison principle for nonlinear second order elliptic operators $F(x,u,\nabla u,\nabla^2 u)$ is the following. Here $F(x,s,p,M)$ is of class C^1, $x\in\Omega$, $s\in\mathbb{R}^1$, $p\in\mathbb{R}^n$, $M\in\mathcal{S}^{n\times n}$, the set of all $n\times n$ real symmetric matrices, and is elliptic, i.e.,

$$\frac{\partial F}{\partial M_{ij}}>0.$$

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Strong comparison principle. Let $u,v\in C^2(\Omega)$ satisfy (1.1) and

$$F(x,u,\nabla u,\nabla^2 u)\le F(x,v,\nabla v,\nabla^2 v) \quad \text{in} \quad \Omega.$$

Then we have

$$\text{either} \quad u>v \quad \text{in} \quad \Omega, \quad \text{or} \quad u\equiv v \quad \text{in} \quad \Omega.$$

In [9] and [10], L. Caffarelli, L. Nirenberg and the first named author obtained some forms of the comparison principle for singular solutions of the nonlinear elliptic operators of the form $F(x,u,\nabla u,\nabla^2 u)$. Recently, the strong comparison principle and the Hopf Lemma was established for viscosity solutions to the equations of the form $F(x,u,\nabla u,\nabla^2 u)=1$ when one of the competitors is $C^{1,1}$ by Y. Y. Li, L. Nguyen and B. Wang in [33].

In recent years, comparison principles for degenerate elliptic equations have been widely studied; see [1-7, 14-37] and the references therein. One type of those equations, which appeared in [9-12,18,19,29,30,34,35], involve a symmetric matrix function

$$G[u]:=\nabla^2 u+L(x,u,\nabla u), \tag{1.2}$$

where $L\in C^{0,1}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, is in $\mathcal{S}^{n\times n}$.

One such matrix operator is the conformal Hessian matrix operator (see e.g., [28, 38] and the references therein), i.e.,

$$H[u]=\nabla^2 u+\nabla u\otimes\nabla u-\frac{1}{2}|\nabla u|^2 I_n,$$

where I_n denotes the $n\times n$ identity matrix, and, for $p,q\in\mathbb{R}^n$, $p\otimes q$ denotes the $n\times n$ matrix with entries $(p\otimes q)_{ij}=p_iq_j$ for $i,j=1,\cdots,n$.

Let U be an open subset of $\mathcal{S}^{n\times n}$ satisfying

$$0\in\partial U, \quad U+\mathcal{P}\subset U, \quad O^tUO\subset U, \quad \forall O\in O(n) \quad tU\subset U, \quad \forall t>0,$$

where $\mathcal{P}$ denotes the set of all $n\times n$ non-negative real matrices and $O(n)$ denotes the set of all $n\times n$ real orthogonal matrices.

In [34], it was shown that, under the assumption

$$\text{diag}\{1,0,\cdots,0\}\in\partial U, \tag{1.3}$$

the strong comparison principle and Hopf Lemma fail for a class of equations of the form

$$G[u]\in\partial U.$$

Conversely, if (1.3) does not hold, then the strong comparison principle and Hopf Lemma holds.

Although the strong comparison principle fails under assumption (1.3), the first named author proved that a weak form of strong comparison principle holds for the