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Smulyan Lemma and Differentiability of the Support Function

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Abstract. The purpose of this paper is to verify the Smulyan lemma for the support function, and also the Gateaux differentiability of the support function is studied on its domain. Moreover, we provide a characterization of Frechet differentiability of the support function on the extremal points.

Key Words: Frechet and Gateaux differentiability, support function, strict convexity, Smulyan lemma.

AMS Subject Classifications: 49j50, 52A05, 52A41

1 Introduction

The problem of differentiability and subdifferentiability of a convex function on a Banach space X are important in the theory of optimization (specially in economics) and geometry of Banach spaces. Recently, this issue has been discussed for specific convex functions known as support functions. In fact, they play a fundamental role in the development of optimization and variational analysis.

In economics, maximization of linear functionals on the subsets of Banach spaces has special importance in optimizing the price and profit. Shephard's lemma is one of the most important results in economics. It is also associated with the differentiability of the cost function (see [10]) defined by

$$g:\mathbb{R}^p_+\to\bar{\mathbb{R}},\quad g(x):=\inf_{a\in A}x(a),$$

where p is a positive integer, $\mathbb{R}^p$ is the p-dimensional Euclidean space and A is a subset of $\mathbb{R}^p_+$ (the positive cone of $\mathbb{R}^p$).

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Let X be a Banach space and A be a subset of X. Support function of the set A is defined by

$$\sigma_A : X^* \to \bar{\mathbb{R}}, \quad \sigma_A(x^*) := \sup_{t\in A} x^*(t).$$

Clearly, when $\dim X = p$, the cost function g is strongly related to the support function σ_A. In fact, any property of the support function σ_A can be translated to a corresponding property of the cost function. See [11,12] and the long list of references therein.

This article is organized as follows. In Section 2, we present some preliminaries. In Section 3, we state Smulyan lemma forthe support function and we establish some results regarding Smulyan lemma on the Gateaux and Frechet differentiability of support function. In Section 4, we show that the support function σ_A is Gateaux differentiable on the interior of its domain $\mathrm{int}(\mathrm{dom}\sigma_A)$, which is an extension of [11, Theorem 6] into the infinite dimensional case.

2 Preliminaries

Throughout this paper, $(X, \|\cdot\|)$ is a real Banach space whose dual X^* is endowed with the dual norm, denoted also by $\|\cdot\|$. We consider $A \subset X$ a nonempty set. As usual, we denote the interior of A, the cone generated by A, the affine hull of A, the linear space parallel to $\mathrm{aff}A$, the relative interior of A (that is the interior of A with respect to affine hull of A), the relative boundary of A, the closure of A, the convex hull of A and the polar set of A by $\mathrm{int}A$, $\mathrm{cone}A$, $\mathrm{aff}A$, $\mathrm{lin}_0 A$, $\mathrm{rint}A$, $\mathrm{rbd}A$, $\mathrm{cl}A$, $\mathrm{conv}A$ and A^0, respectively.

Let U be an open subset of the Banach space X and $f: U \to \mathbb{R}$ be a real valued function. We say that f is Gateaux differentiable at $x \in U$, if for every $h \in X$,

$$f'(x)(h) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}$$

exists in $\mathbb{R}$ and $f'(x)$ is a linear continuous function at h (i.e., $f'(x) \in X^*$). The functional $f'(x)$ is then called the Gateaux derivative or Gateaux differential of f at x. If, in addition, the above limit is uniform at $h \in S_X$ (where S_X denotes the unit sphere in X), we say that f is Frechet differentiable at x. See [4] for more details.

We recall that the domain of a convex extended-valued function $f: X \to \bar{\mathbb{R}}$ is the set

$$\mathrm{dom} f := \{x \in X : f(x) < \infty\}.$$

A convex extended-valued function f is proper if and only if $\mathrm{dom} f \neq \emptyset$ and $f(x) \neq -\infty$ for each $x \in X$ [1]. The subdifferential of a proper function f at $x \in \mathrm{dom} f$ is

$$\partial f(x) := \{x^* \in X^* : x^*(y-x) \leq f(y) - f(x), \ \forall y \in X\},$$

and the domain of ∂f is defined by

$$\mathrm{dom}\partial f = \{x \in X : \partial f(x) \neq \emptyset\} (\subset \mathrm{dom} f),$$