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A Note on a Theorem of T. J. Rivlin

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Abstract. In this paper, we obtain a result that improves the results of Govil and Nwaeze, Qazi and the classical result of Rivlin.

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1 Introduction and statement of results

For an arbitrary entire function $f(z)$, let $M(f,r)=\max_{|z|=r}|f(z)|$. For a polynomial $P(z)=\sum_{j=0}^{n}a_jz^j$ of degree n, it is known that

$$M(P,r)\geq r^nM(P,1), \quad r\leq 1. \tag{1.1}$$

Inequality (1.1) is due to Varga [7] who attributed it to Zarantonello.

It is noted that equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin, so it is natural to seek improvement under appropriate assumption on the zeros of $P(z)$. It was shown by Rivlin [6] that if $P(z)\neq 0$ in $|z|<1$, then (1.1) can be replaced by

$$M(P,r)\geq\left(\frac{1+r}{2}\right)^nM(P,1) \quad \text{for } r\leq 1. \tag{1.2}$$

As a generalization of (1.2), Govil [2] proved that if $P(z)\neq 0$ in $|z|<1$, then for $0<r\leq R\leq 1$,

$$M(P,r)\geq\left(\frac{1+r}{1+R}\right)^nM(P,R). \tag{1.3}$$

In 1992, Qazi [4] generalized (1.3) in a different direction and proved that if $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu<n$ is a polynomial of degree n not vanishing in $|z|<1$ then for $0<r<R\leq 1$,

$$M(P,r)\geq\left(\frac{1+r^\mu}{1+R^\mu}\right)^{\frac{n}{\mu}}M(P,R). \tag{1.4}$$

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More recently, Govil and Nwaeze [3] besides proving some other results, also proved the following generalization and refinement of (1.3).

Theorem 1.1. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$. If $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then for $0<r<R\leq 1$,*

$$M(P,r)\geq \frac{(1+r)^n}{(1+r)^n+(R+k)^n-(r+k)^n}\left\{M(P,R)+m\ln\left(\frac{R+k}{r+k}\right)^n\right\}, \tag{1.5}$$

where $m=\min_{|z|=k}|P(z)|$.

Some more results related to inequalities that compares the growth of a polynomial on $|z|=r$ and $|z|=R$, where $r<R$, can be found in (see [4,8]).

In this note, we present the following extension of Theorem 1.1. As we shall see our result provides refinements of (1.2), (1.3) and (1.4) as well.

Theorem 1.2. *Let $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu<n$. If $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then for $0<r<R\leq 1$,*

$$M(P,r)\geq \frac{(1+r^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}}+(R^\mu+k^\mu)^{\frac{n}{\mu}}-(r^\mu+k^\mu)^{\frac{n}{\mu}}}\left\{M(P,R)+m\ln\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}\right\}, \tag{1.6}$$

where $m=\min_{|z|=k}|P(z)|$.

Remark 1.1. For $\mu=1$, Theorem 1.2 reduces to Theorem 1.1. Taking $k=1$ in Theorem 1.2 we get the following refinement of (1.4).

Corollary 1.1. Let $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu<n$. If $P(z)\neq 0$ in $|z|<1$, then for $0<r<R\leq 1$,

$$M(P,r)\geq\left(\frac{1+r^\mu}{1+R^\mu}\right)^{\frac{n}{\mu}}\left\{M(P,R)+m\ln\left(\frac{1+R^\mu}{1+r^\mu}\right)^{\frac{n}{\mu}}\right\},$$

where $m=\min_{|z|=1}|P(z)|$.

If we take $R=1$ in Theorem 1.2, we get

Corollary 1.2. Let $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu<n$. If $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then for $0<r<1$,

$$M(P,r)\geq \frac{(1+r^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}}+(1+k^\mu)^{\frac{n}{\mu}}-(r^\mu+k^\mu)^{\frac{n}{\mu}}}\left\{M(P,1)+m\ln\left(\frac{1+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}\right\},$$

where $m=\min_{|z|=k}|P(z)|$.

The following extension and refinement of inequality (1.2) due to Rivlin [6] immediately follows from Corollary 1.2 by taking $k=1$ in it.

Corollary 1.3. Let $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu<n$. If $P(z)\neq 0$ in $|z|<1$, then for $0<r<1$,

$$M(P,r)\geq\left(\frac{1+r^\mu}{2}\right)^{\frac{n}{\mu}}\left\{M(P,1)+m\ln\left(\frac{2}{1+r^\mu}\right)^{\frac{n}{\mu}}\right\},$$

where $m=\min_{|z|=1}|P(z)|$.