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Existence of Solutions for Fractional Differential Equations Involving Two Riemann-Liouville Fractional Orders

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Abstract. In this work, we study existence and uniqueness of solutions for multi-point boundary value problem of nonlinear fractional differential equations with two fractional derivatives. By using the variety of fixed point theorems, such as Banach's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder's degree theory, the existence of solutions is obtained. At the end, some illustrative examples are discussed.

Key Words: Riemann-Liouville integral, existence, fixed point theorem, Leray-Shauders alternative.

AMS Subject Classifications: 26A33, 34A12, 34A08

1 Introduction

Fractional derivative arises from many physical processes, such as a charge transport in amorphous semiconductors [22], electrochemistry and material science, they are in fact described by differential equations of fractional order [9, 10, 17, 18]. Recently, many studies on fractional differential equations, involving different operators such as Riemann-Liouville operators [19, 24], Caputo operators [1, 3, 13, 25], Hadamard operators [23] and $q-$fractional operators [2], have appeared during the past several years. Moreover, by applying different techniques of nonlinear analysis, many authors have obtained results of the existence and uniqueness of solutions for various classes of fractional differential equations, for example, we refer the reader to [3–8, 11, 12, 14, 15, 19] and the references cited therein.

In this work, we discuss the existence and uniqueness of the solutions for multi-point boundary value problem of nonlinear fractional differential equations with two

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Riemann-Liouville fractional orders

$$\begin{cases} D^{\alpha_1}(D^{\alpha_2}+\lambda)x(t)=f(t,x(t))+\sum\limits_{i=1}^{m}A_iJ^{\beta_i}g_i(t,x(t)), & t\in[0,T],\\ J^{1-\alpha_2}x(0)=0, \quad D^{\alpha_2+\alpha_1-2}x(T)=\sum\limits_{j=1}^{k}B_jJ^{\alpha_2+\alpha_1-1}x(\eta_j), & 0<\eta_j<T, \end{cases} \tag{1.1}$$

where D^{α_l} is the Riemann-Liouville fractional derivative of order α_l, with $0<\alpha_l\le 1$, $(l=1,2)$, $1<\alpha_1+\alpha_2\le 2$, J^{ϑ} is the Riemann-Liouville fractional integral of order $\vartheta>0$, $\vartheta\in\{\beta_i,1-\alpha_2,\alpha_2+\alpha_1-1\}$, λ, A_i, B_j are real constants and f, $g_i:[0,T]\times\mathbb{R}\to\mathbb{R}$, $1\le i\le m$, $1\le j\le k$, $k\ge 2$ are continuous functions on $[0,T]$.

The existence results for the multi-point boundary value problem (1.1) are based on variety of fixed point theorems, such as Banach's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder's degree theory.

2 Preliminaries

In this section, we present notation and some preliminary lemmas that will be used in the proofs of the main results.

Definition 2.1 (see [20,21])**.** The Riemann-Liouville fractional integral of order $\vartheta\ge 0$, of a function $h:(0,\infty)\to\mathbb{R}$ is defined as

$$J^{\vartheta}h(t)=\frac{1}{\Gamma(\vartheta)}\int_0^t(t-\tau)^{\vartheta-1}h(\tau)d\tau,$$
$$J^0h(t)=h(t),$$

where $\Gamma(\vartheta):=\int_0^{\infty}e^{-u}u^{\vartheta-1}du$.

Definition 2.2 (see [20,21])**.** The Riemann-Liouville fractional derivative of order $\vartheta>0$, of a continuous function $h:(0,\infty)\to\mathbb{R}$ is defined as

$$D^{\vartheta}h(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-\tau)^{n-\vartheta-1}h(\tau)d\tau,$$

where $n=[\vartheta]+1$.

For $\vartheta<0$, we use the convention that $D^{\vartheta}h=J^{-\vartheta}h$. Also for $0\le\rho<\vartheta$, it is valid that $D^{\rho}J^{\vartheta}h=h^{\vartheta-\rho}$.

We note that for $\varepsilon>-1$ and $\varepsilon\ne\vartheta-1,\vartheta-2,\cdots,\vartheta-n$, we have

$$D^{\vartheta}t^{\varepsilon}=\frac{\Gamma(\varepsilon+1)}{\Gamma(\varepsilon-\vartheta+1)}t^{\varepsilon-\vartheta},$$
$$D^{\vartheta}t^{\vartheta-i}=0,\quad i=1,2,\cdots,n.$$