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Erdös Type Inequality for Lorentz Polynomials

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Abstract. An elementary, but very useful tool for proving inequalities for polynomials with restricted zeros is the Bernstein or Lorentz representation of polynomials. In the present paper, we give two classes of Lorentz polynomials, for which the Erdös-type inequality holds.

Key Words: Lorentz representation of polynomials, constrained polynomials, Morkov-type inequalities, Erdös-type inequality.

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1 Introduction and main results

Let π_n denote the set of all polynomials of degree at most n with real coefficients, and π_n^c denote the set of all polynomials of degree at most n with complex coefficients, where n is a nonnegative integer.

Let

$$\|f\|_A = \sup_{x\in A} |f(x)|$$

denote the supremum norm of a function f defined on a set A. The Markov inequality is that

$$\|p_n'\|_{[-1,1]} \le n^2 \|p_n\|_{[-1,1]} \tag{1.1}$$

holds for all $p_n \in \pi_n^c$. And the Bernstein inequality is that

$$|p_n'(x)| \le \frac{n}{\sqrt{1-x^2}} \|p_n(x)\|_{[-1,1]} \tag{1.2}$$

holds for all $p_n \in \pi_n^c$ and for all $x \in (-1,1)$. For proofs of these see [2] or [3]. Polynomial inequalities are very basic in several disciplines and they play a foundational role in

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approximation theory. There are a lot of papers devoted to them; see, e.g., [2] and [11]. In [7], Erdös gave a class of restricted polynomials for which the Markov factor n^2 in (1.1) improves to cn. He proved the following theorem.

Theorem 1.1. *The inequality*

$$\|p_n'\|_{[-1,1]} \le \frac{en}{2} \|p_n\|_{[-1,1]} \tag{1.3}$$

holds for all $p_n \in \pi_n$ having all their zeros in $\mathbb{R}\backslash(-1,1)$.

And in [7], Erdös claimed the following theorem but did not give a hint to prove it. Recently, Erdélyi proved it in [5]. Using different approach, We gave another proof of Theorem E_2 in [14].

Theorem 1.2. *The inequality*

$$\|p_n'\|_{[-1,1]} \le \frac{n}{2} \|p_n\|_{[-1,1]} \tag{1.4}$$

holds for all $p_n \in \pi_n$ having all their zeros in $\mathbb{R}\backslash(-1,1)$ and $p_n'(x) \neq 0$, $x \in (-1,1)$.

These results motivated several people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. These people include Lorentz [8], Scheick [12], Szabados [13], Máté [10], P. Borwein [1], Erdélyi [4, 5] and others.

An elementary, but very useful tool for proving inequalities for polynomials with restricted zeros is the Bernstein or Lorentz representation of polynomials; see, e.g., [2] and [6].

Let

$$B_n(-1,1) = \left\{ p_n \in \pi_n \,\middle|\, p_n = \sum_{k=0}^{n} a_k (1-x)^k (1+x)^{n-k},\ a_k \ge 0,\ k = 0,1,\cdots,n \right\}. \tag{1.5}$$

And if $p_n(x) \in B_n(-1,1)$ or $-p_n(x) \in B_n(-1,1)$, we call it Lorentz polynomial. As Lorentz observed that, if $p_n(x) \in \pi_n$ having all their zeros outside the open unit disk, then $p_n(x)$ is a Lorentz polynomial.

In this paper, we give two classes of Lorentz polynomials, for which the Erdös-type inequality (1.4) holds. Our main results are the following.

Theorem 1.3. *Let $n \ge 1$ be an integer and $p_n(x) \in \pi_n \backslash \pi_{n-1}$ be a Lorentz polynomial. If $p_n'(x) \in B_{n-1}(-1,1)$ or $-p_n'(x) \in B_{n-1}(-1,1)$, then*

$$\|p_n'\|_{[-1,1]} \le \frac{n}{2} \|p_n\|_{[-1,1]}. \tag{1.6}$$

Equality holds only for $p_n(x) = \sigma c(1+\sigma x)^n$ or $p_n(x) = \sigma c[2^n - (1-\sigma x)^n]$ with a constant $0 \neq c \in \mathbb{R}$ and $\sigma \in \{-1,1\}$.