Analysis in Theory and Applications
Anal. Theory Appl., Vol. **33**, No. 4 (2017), pp. 355-365

DOI: 10.4208/ata.2017.v33.n4.5

Evaluation of Certain Integrals Involving the Product of Classical Hermite's Polynomials Using Laplace Transform Technique and Hypergeometric Approach

M. I. Qureshi and Saima Jabee*

Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025, India

Received 2 June 2017; Accepted (in revised version) 14 August 2017

Abstract. In this paper some novel integrals associated with the product of classical Hermite's polynomials

$$\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_r(x)\}^2dx,\quad \int_0^{\infty}\exp(-x^2)H_{2k}(x)H_{2s+1}(x)dx,$$
$$\int_0^{\infty}\exp(-x^2)H_{2k}(x)H_{2s}(x)dx\quad \text{and}\quad \int_0^{\infty}\exp(-x^2)H_{2k+1}(x)H_{2s+1}(x)dx,$$

are evaluated using hypergeometric approach and Laplace transform method, which is a different approach from the approaches given by the other authors in the field of special functions. Also the results may be of significant nature, and may yield numerous other interesting integrals involving the product of classical Hermite's polynomials by suitable simplifications of arbitrary parameters.

Key Words: Gauss's summation theorem, classical Hermite's polynomials, generalized hypergeometric function, generalized Laguerre's polynomials.

AMS Subject Classifications: 33C20, 33C45, 33C47

1 Introduction

In recent years, numerous integral formulae involving a variety of special functions have been established by many authors (see, [1–5,7]). Also many integral formulae associated with the general class of polynomials (Laguerre, Hermite, Legendre, Bessel, Tchebychev and as in Askey-scheme) and other special cases therein (see, [6,8–12]).

Many integral formulae involving the products of classical orthogonal polynomials have been developed and play an important role in several physical problems. In fact,

*Corresponding author. *Email addresses:* `miqureshi_delhi@yahoo.co.in` (M. I. Qureshi), `saimajabee007@gmail.com` (S. Jabee)

Hermite's polynomials are associated with a wide range of problems in diverse areas of mathematics. These connections of Hermite's polynomials with various other research areas have led many researchers to the field of special functions. we aim at presenting four integral formulae involving the product of classical polynomials of Hermite using Laplace transform method and hypergeometric approach. Among many properties of Hermite's polynomials, they also have investigated some possible extensions of the Hermite's polynomials. Those integrals involving the general class of polynomials are not only of great interest in pure mathematics, but they are often of extreme importance in many branches of theoretical and applied physics and engineering.

The widely-used Pochhammer symbol $(\lambda)_\nu$, $(\lambda,\nu\in\mathbb{C})$ is defined by

$$(\lambda)_\nu:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1, & (\nu=0;\ \lambda\in\mathbb{C}\backslash\{0\}),\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases} \tag{1.1}$$

it being understood *conventionally* that $(0)_0=1$ and assumed *tacitly* that the Γ quotient exists.

The *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1,\alpha_2,\cdots,\alpha_p$ and q denominator parameters $\beta_1,\beta_2,\cdots,\beta_q$, is defined by

$${}_pF_q\left[\begin{array}{l}\alpha_1,\alpha_2,\cdots,\alpha_p;\\ \beta_1,\beta_2,\cdots,\beta_q;\end{array}\ z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1.2}$$

$(p,q\in\mathbb{N}_0;\ p\leqq q+1;\ p\leqq q$ and $|z|<\infty$, $p=q+1$ and $|z|<1$; $p=q+1$, $|z|=1$, and $\Re(\omega)>0$; $p=q+1$, $|z|=1$, $z\neq 1$ and $0\geq\Re(\omega)>-1)$, where

$$\omega:=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j,\quad \big(\alpha_j\in\mathbb{C},\quad (j=1,2,\cdots,p);\quad \beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-\,(j=1,2,\cdots,q)\big).$$

Laplace transform of $t^{\alpha-1}$:

$$\int_0^\infty e^{-st}t^{\alpha-1}dt=\frac{\Gamma(\alpha)}{s^\alpha}, \tag{1.3}$$

where $\Re(s)>0$, $0<\Re(\alpha)<\infty$ or $\Re(s)=0$, $0<\Re(\alpha)<1$.

Guass's summation theorem: The Gauss's summation theorem plays a vital role in the proof of many interesting results and some physical problems [13, pp. 49(Theorem 18)]

$${}_2F_1\left[\begin{array}{c}a,b;\\ c;\end{array}\ 1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.4}$$

where $c\neq 0,-1,-2,-3,\cdots$, and $\Re(c-a-b)>0$.

Special case of Gauss's summation theorem: By using the Gauss's summation theorem, it is easy to prove (see, [13, pp. 69(Q.N.4)])

$${}_2F_1\left[\begin{array}{c}-n,b;\\ c;\end{array}\ 1\right]=\frac{(c-b)_n}{(c)_n}, \tag{1.5}$$