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Parameterized Littlewood-Paley Operators on Weighted Herz Spaces

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Abstract. The strong type and weak type estimates of parameterized Littlewood-Paley operators on the weighted Herz spaces $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ are considered. The boundedness of the commutators generated by BMO functions and parameterized Littlewood-Paley operators are also obtained.

Key Words: Parameterized Littlewood-Paley operator, Herz space, weak Herz space, BMO, commutator, Muckenhoupt weight.

AMS Subject Classifications: 42B20, 30H35, 42B35

1 Introduction

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n (n\geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega\in L^1(\mathbb{S}^{n-1})$ and

$$\int_{\mathbb{S}^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.1}$$

where $x'=x/|x|$ for any $x\neq 0$. Then for $0<\rho<n$, the area integral $\mu^{\rho}_{\Omega,S}$ and the Littlewood-Paley $\mu^{*,\rho}_{\lambda}$- function are defined respectively by

$$\mu^{\rho}_{\Omega,S}(f)(x)=\left(\iint_{\Gamma(x)}\left|\frac{1}{t^{\rho}}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

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and

$$\mu_\lambda^{*,\rho}(f)(x)=\left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\lambda>1$ and $\Gamma(x)=\{(y,t)\in\mathbb{R}_+^{n+1}:\ |x-y|<t\}$.

Now let us turn to the introductions of the corresponding commutators of the parameterized Littlewood-Paley operators above. Let $b\in L^1_{loc}(\mathbb{R}^n)$, $m\in\mathbb{N}$, the commutators $[b^m,\mu^\rho_{\Omega,S}]$ and $[b^m,\mu_\lambda^{*,\rho}]$ are defined respectively by

$$\begin{aligned}&[b^m,\mu^\rho_{\Omega,S}](f)(x)\\=&\left(\iint_{\Gamma(x)}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}&[b^m,\mu_\lambda^{*,\rho}](f)(x)\\=&\left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}.\end{aligned}$$

In 1990, Torchinsky and Wang [1] gave the weighted $L^2(\mathbb{R}^n)$ boundedness of $\mu^\rho_{\Omega,S}$ and $\mu_\lambda^{*,\rho}$ for $\rho=1$ and $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})$ $(0<\alpha\le 1)$. Here, we say that $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})$ if

$$|\Omega(x')-\Omega(y')|\le|x'-y'|^\alpha,\quad x',y'\in\mathbb{S}^{n-1}. \tag{1.2}$$

For general ρ, in 1999, Sakamoto and Yabuta [2] gave $L^p(\mathbb{R}^n)$ boundedness for $\mu^\rho_{\Omega,S}$ and $\mu_\lambda^{*,\rho}$ when $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})$.

Suppose that $\Omega\in L^q(\mathbb{S}^{n-1})$, $q\ge 1$. Then the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta)=\sup_{\|\gamma\|\le\delta}\left(\int_{\mathbb{S}^{n-1}}|\Omega(\gamma x')-\Omega(x')|^q d\sigma(x')\right)^{1/q},$$

where γ denotes a rotation on $\mathbb{S}^{n-1}$ and $\|\gamma\|=\sup_{x'\in\mathbb{S}^{n-1}}|\gamma x'-x'|$.

Recently, Ding and Xue obtained the following weighted results.

Theorem 1.1 (see [3]). *Suppose $\rho>n/2$, $\lambda>2$ and $\Omega\in L^2(\mathbb{S}^{n-1})$ satisfies*

$$\int_0^1\frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma<\infty \tag{1.3}$$

for some $\sigma>1$. If $1<p<\infty$ and $\omega\in A_p$, then both of $\mu^\rho_{\Omega,S}$ and $\mu_\lambda^{*,\rho}$ are bounded on the weighted space $L^p(\mathbb{R}^n,\omega)$.